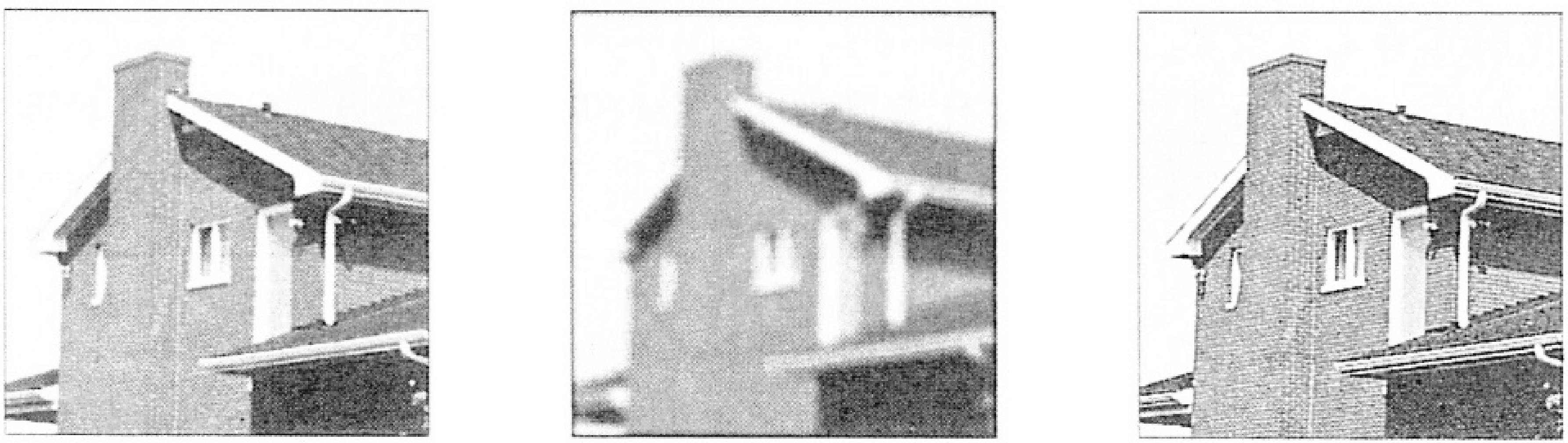
Advanced Information Engineering

#10 December 14 (Mon), 2020

Kenjiro T. Miura

Filtering in Space Domain

- Perform multi-dimensional filtering represented by a linear shift-invariant system.



(a) 原画像　(b) 処理例 1　(c) 処理例 2

図 3.6　フィルタによる画像処理例

2D Output System

$$y(n_1, n_2) = T[x(n_1, n_2)]$$

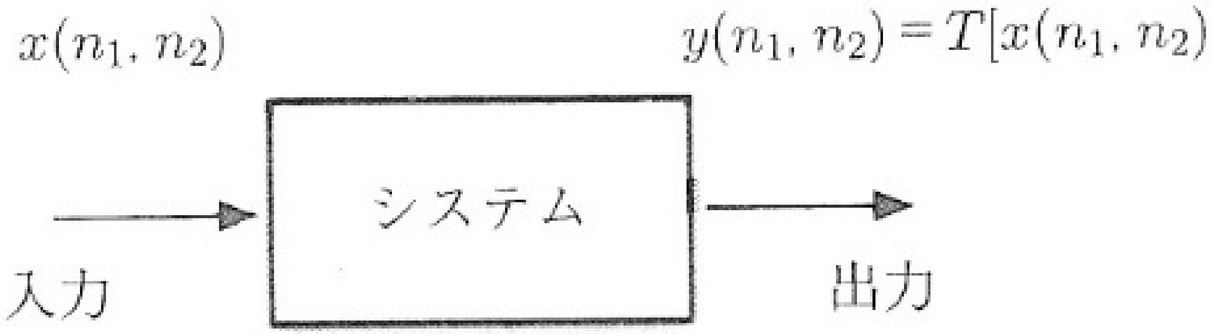


図 **3.7**　2 次元システム

Filtering Example in Space Domain

- Moving average filter : by changing the center pixel, calculate average of 3×3 pixels.

$$y(n_1, n_2) = \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2)$$

$$= \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} 1/9 \cdot x(n_1 - k_1, n_2 - k_2)$$

Moving Average Filter

- Weight *h* is called filter matrix or impulse response.

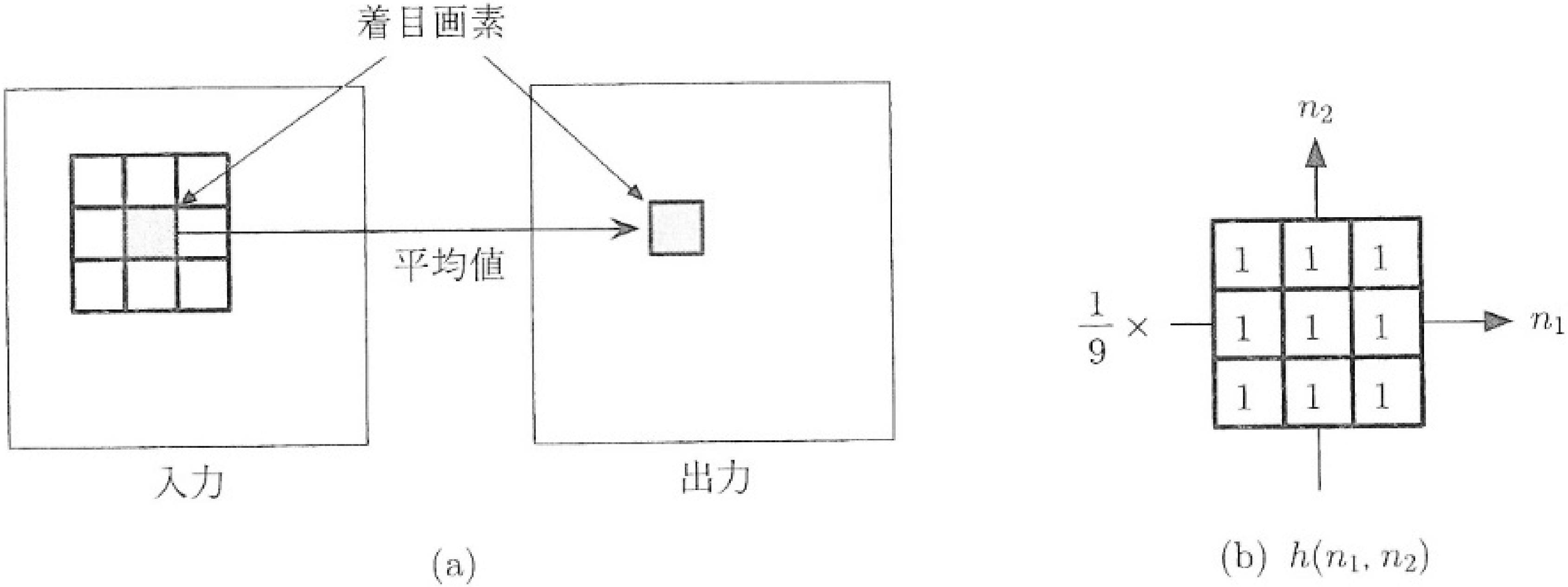


図 3.8 空間領域におけるフィルタ処理

Impulse Response

- Basic response of the linear shift-invariant system.

$$h(n_1, n_2) = T[\delta(n_1, n_2)]$$

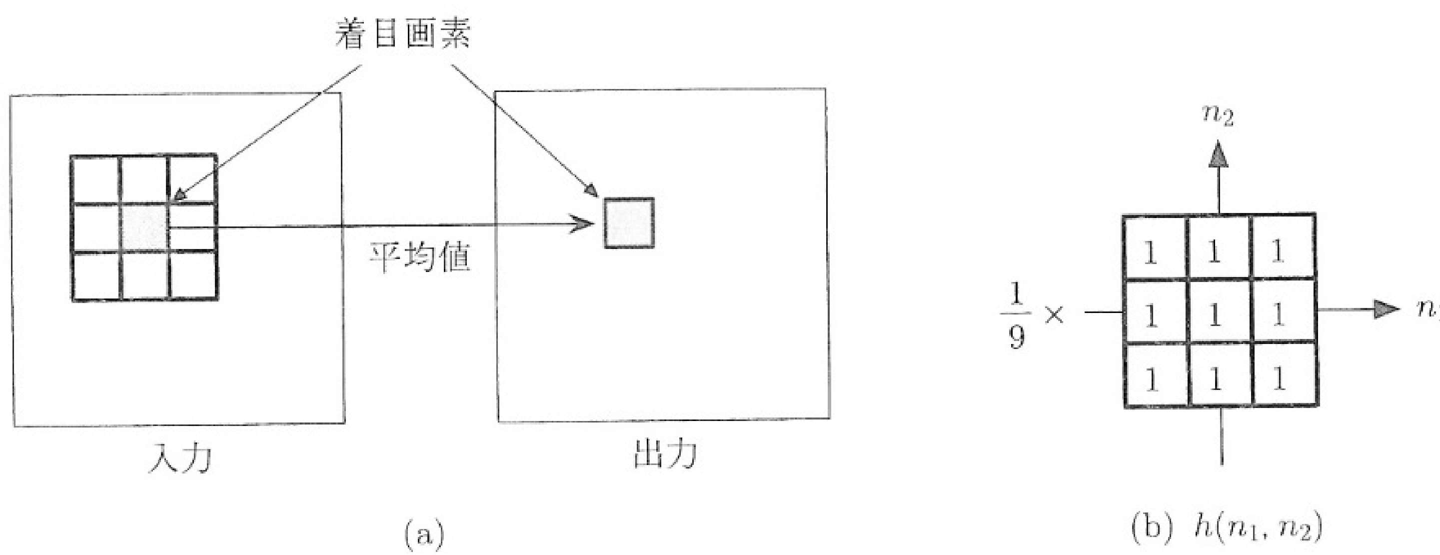


図 3.8　空間領域におけるフィルタ処理

Shift-invariant System

- The following equation is satisfied for arbitrary input.
- Note that k_1 and k_2 are arbitrary integers.

$$y(n_1 - k_1, n_2 - k_2) = T[x(n_1 - k_1, n_2 - k_2)]$$

Linear System

- The following equation is satisfied for arbitrary input.
- Note that k_1 and k_2 are arbitrary integers.

$$T[ax_1(n_1, n_2) + bx_2(n_1, n_2)] = ay_1(n_1, n_2) + by_2(n_1, n_2)$$

Linear shift-invariant system

- System which satisfies linearity and shift - invariance.
- What kind of response of (a) ?
- Show (b) and (c) ?

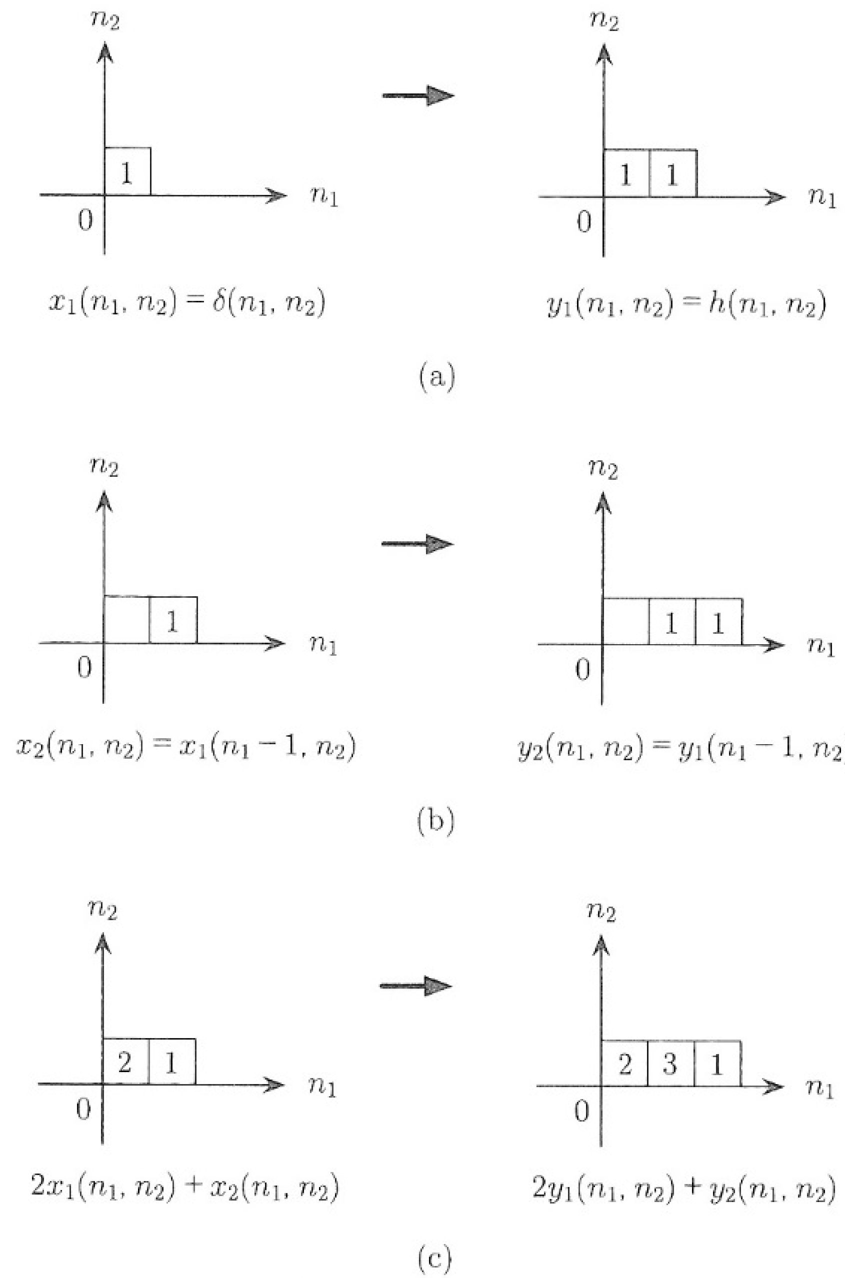


図 **3.9** 線形性とシフト不変性

Convolution

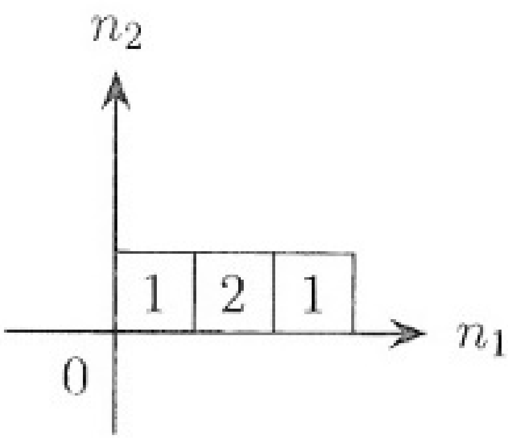
- For a linear shift-invariant system its output is calculated as a weighted sum (convolution).

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2)$$

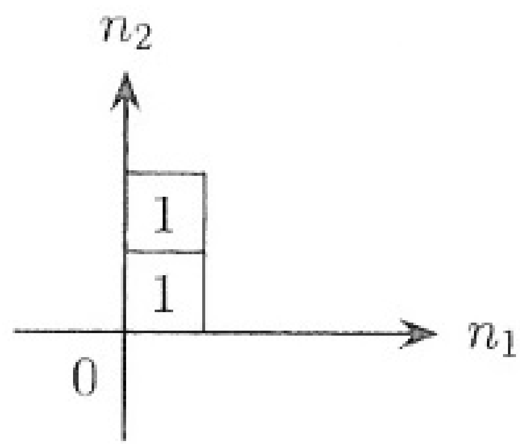
$$= h(n_1, n_2) * x(n_1, n_2)$$

Exercise Example

- Under the assumption of linear shift-invariance, perform convolution by using the input signal in Fig.(a) and impulse $h(n_1,n_2)$.



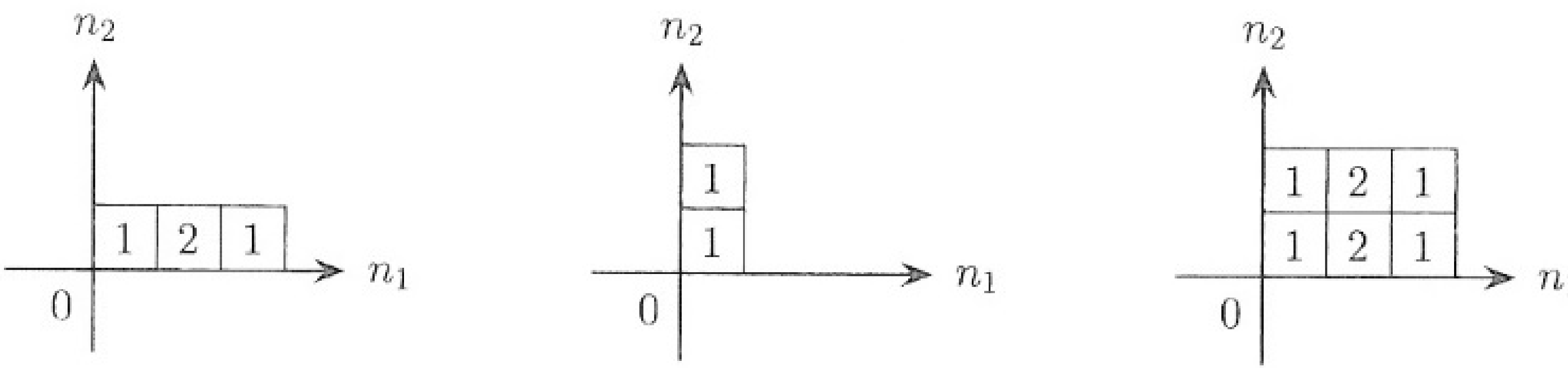
(a) $x(n_1, n_2)$



(b) $h(n_1, n_2)$

Answer

$$
\begin{aligned}
y(n_1, n_2) &= \sum_{k_1=0}^{2} \sum_{k_2=0}^{0} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2) \\
&= h(n_1, n_2) + 2h(n_1 - 1, n_2) + h(n_1 - 2, n_2)
\end{aligned}
$$



(a) $x(n_1, n_2)$ (b) $h(n_1, n_2)$ (c) $y(n_1, n_2)$

Exercise Examples

- Show the impulse response for the following systems.

【例題 3.6】 以下のシステムのインパルス応答を示せ.

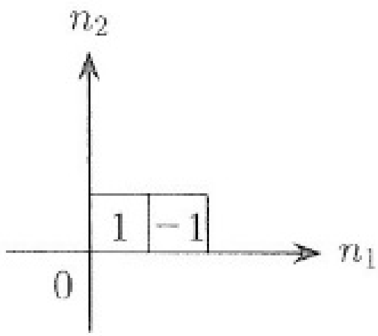
(a) $y(n_1, n_2) = x(n_1, n_2) - x(n_1 - 1, n_2)$

(b) $y(n_1, n_2) = x(n_1, n_2) - x(n_1, n_2 - 1)$

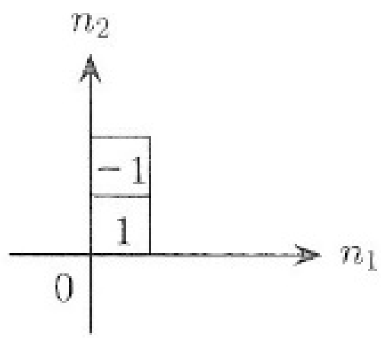
(c) $y(n_1, n_2) = \{x(n_1, n_2) + x(n_1 - 1, n_2) + x(n_1, n_2 - 1)$
$+x(n_1 - 1, n_2 - 1)\}/4$

Answers

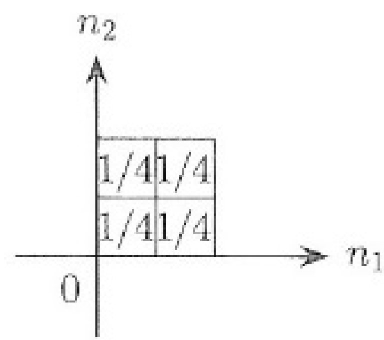
【解答】 (a) $x(n_1, n_2) = \delta(n_1, n_2)$ を代入すると，$h(n_1, n_2) = \delta(n_1, n_2) - \delta(n_1 - 1, n_2)$ となる (図 3.12(a) 参照)．同様に考えると，(b)(c) に対して図 3.12(b)(c) を得る． □



(a)



(b)



(c)

Frequency Characteristic of System

- One of the purposes of filtering is to pass specific frequency components.
- The filter of a linear shift-invariant system is evaluated by frequency characteristic.

Frequency Characteristic

- Since it is the same equation for discrete spatial Fourier transform, if the signal is even symmetric, the frequency characteristic is real-valued.

$$
\begin{aligned}
H(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)} \\
&= A(\omega_1, \omega_2) e^{j\theta(\omega_1, \omega_2)}
\end{aligned}
$$

Derivation of Frequency Characteristic

- For a linear shift-invariant system, input a complex sinusoidal signal, then….

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2) e^{j(\omega_1(n_1-k_1)+\omega_2(n_2-k_2))}$$

$$= \left(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2) e^{-j(\omega_1 k_1+\omega_2 k_2)} \right) e^{j(\omega_1 n_1+\omega_2 n_2)}$$

$$\begin{aligned} y(n_1, n_2) &= H(e^{j\omega_1}, e^{j\omega_2}) e^{j(\omega_1 n_1+\omega_2 n_2)} \\ &= A(\omega_1, \omega_2) e^{j\theta(\omega_1, \omega_2)} e^{j(\omega_1 n_1+\omega_2 n_2)} \\ &= A(\omega_1, \omega_2) e^{j((\omega_1 n_1+\omega_2 n_2)+\theta(\omega_1, \omega_2))} \end{aligned}$$

Frequency Characteristic

- A linear shift-invariant system outputs sinusoidal wave with the same frequency as the input wave.

- The differences between the input and output are only $A(w_1,w_2)$ and $\theta(w_1,w_2)$.

- $A(w_1,w_2)$ and $\theta(w_1,w_2)$ are obtained from Eq.(3.33) by using $h(n_1,n_2)$.

Exercise Example

- Find the frequency characteristics ($A(w_1,w_2)$ and $\theta(w_1,w_2)$) of the following linear shift-invariant.
- Illustrate the frequency characteristics.

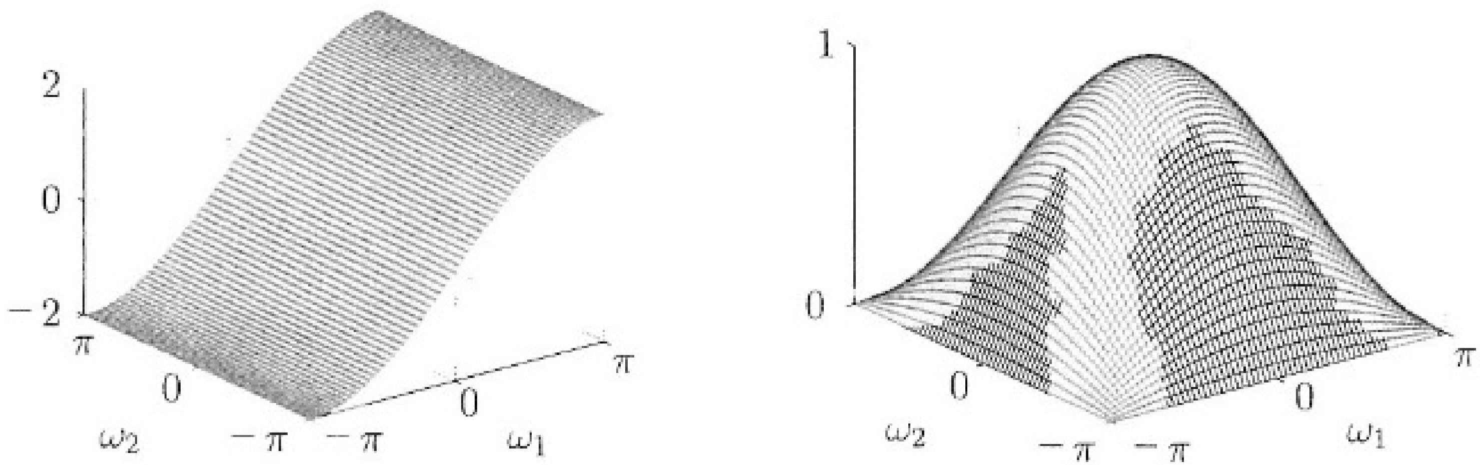
(a) $y(n_1,n_2) = x(n_1,n_2) - x(n_1-1,n_2)$

(b) $y(n_1,n_2) = x(n_1,n_2) - x(n_1,n_2-1)$

(c) $y(n_1,n_2) = \{x(n_1,n_2) + x(n_1-1,n_2) + x(n_1,n_2-1)$
$$+x(n_1-1,n_2-1)\}/4$$

Answer

【解答】 (a) 式 (3.33) にインパルス応答を代入すると，$H(e^{j\omega_1}, e^{j\omega_2}) = 1 - e^{-j\omega_1} = (e^{j\omega_1/2} - e^{j\omega_1/2})e^{-j\omega_1/2} = 2\sin(\omega_1/2)e^{-j(\omega_1/2-\pi/2)}$, $A(\omega_1, \omega_2) = 2\sin(\omega_1/2)$, $\theta(\omega_1, \omega$ $= -(\omega_1/2-\pi/2)$ (c) $H(e^{j\omega_1}, e^{j\omega_2}) = \cos(\omega_1/2)\cos(\omega_2/2)e^{-j(\omega_1/2+\omega_2/2)}$, $A(\omega_1, \omega_2) = \cos(\omega_1/2)\cos(\omega_2/2)$, $\theta(\omega_1, \omega_2) = -(\omega_1/2 + \omega_2/2)$ となる．図 3.14 にこれらの振幅特性 $A(\omega_1, \omega_2)$ を図示する． □



(a) 例題 3.6(a)

(b) 例題 3.6(c)

図 **3.14** 例題 3.7

Z Transform and Transfer Function

- To represent a linear shift-invariant system compactly, we define a transfer function based on Z transform.

2 次元信号 $g(n_1, n_2)$ に対する z 変換は,

$$G(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$$

Z Transform

- Transform defined in discrete space（Laplace transform）
- By substituting Z_1, Z_2 with $e^{j\omega_1}$ and $e^{j\omega_2}$を, we will obtain discrete space Fourier transform.

2 次元信号 $g(n_1, n_2)$ に対する z 変換は,

$$G(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$$

Properties of Z Transform

- Linearity, separability for separable signals, convolution, shift of signals
- Similar properties with Fourier transform

性質	信号	変換領域
線形性	$a_1 g_1(n_1, n_2) + a_2 g_2(n_1, n_2)$	$a_1 G_1(z_1, z_2) + a_2 G_2(z_1, z_2)$
可分性	$g_1(n_1) g_2(n_2)$	$G_1(z_1) G_2(z_2)$
たたみ込み	$h(n_1, n_2) * x(n_1, n_2)$	$H(z_1, z_2) X(z_1, z_2)$
シフト	$g(n_1 - k_1, n_2 - k_2)$	$G(z_1, z_2) z_1^{-k_1} z_2^{-k_2}$

Transfer Function

- Convolution

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2)$$
$$= h(n_1, n_2) * x(n_1, n_2)$$

- From the properties of Z transform

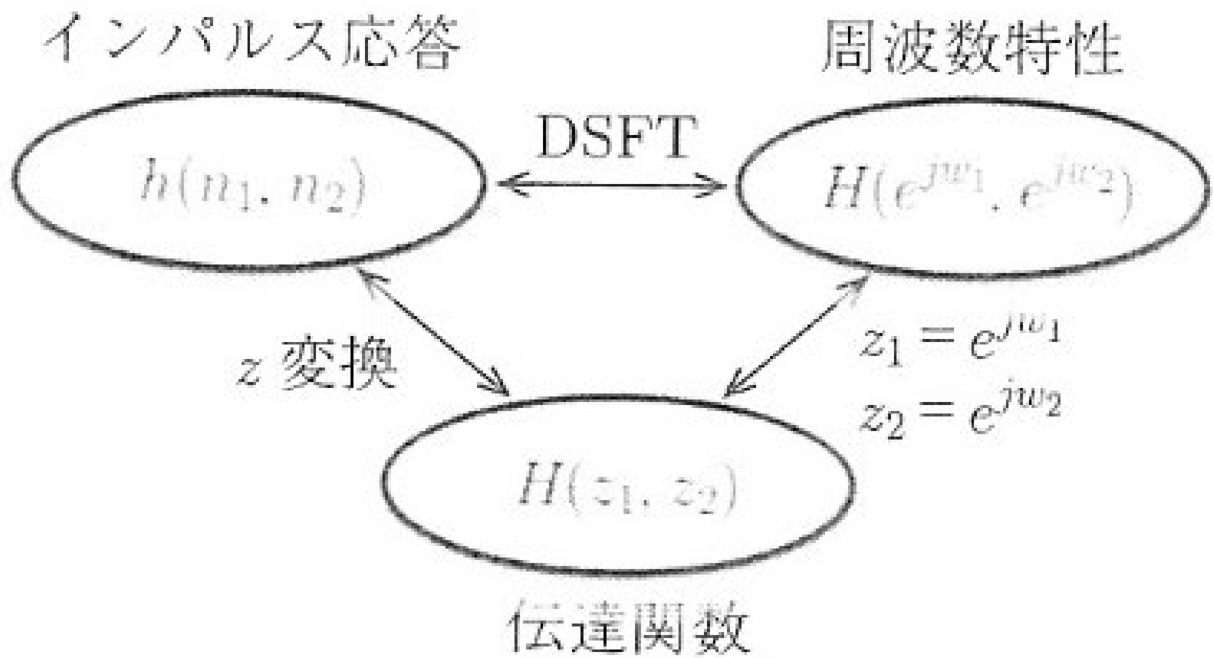
$$Y(z_1, z_2) = H(z_1, z_2) X(z_1, z_2)$$

where $H(z_1,z_2)$ is called a transfer function.

- Ex. Point spread function of optical microscopy and optical transfer function

Transfer Function

- The frequency characteristic is obtained from its transfer function.
- When the impulse response is a separable signal, its filter becomes a separable filter because of the properties of Z transform.



Exercise Example

- Find the transfer function of the following linear shift-invariant system.
- Show that the system is separable.

$$\text{(c) } y(n_1, n_2) = \{x(n_1, n_2) + x(n_1 - 1, n_2) + x(n_1, n_2 - 1) + x(n_1 - 1, n_2 - 1)\}/4$$

Answer

- Input an unit impulse signal to the system and obtain the impulse response.
- Perform Z transform for the impulse response and get the transfer function.

$$H(z_1, z_2) = 1/4 \cdot (1 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1}) = 1/4 \cdot (1 + z_1^{-1})(1 + z_2^{-1})$$

Separable Filter

- Most of the image processing are performed based on separable filters and have the following properties.
- Multi-dimensional filter can consist of iterations of 1D filter processing.
- Its computational cost is relatively low.
- There are limits on realizable frequency characteristics.

Separable Filter

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- Its computational cost is relatively low.

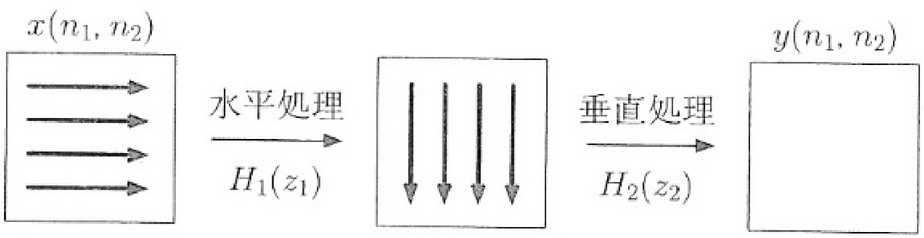

図 **3.17**　分離型フィルタの実行手順

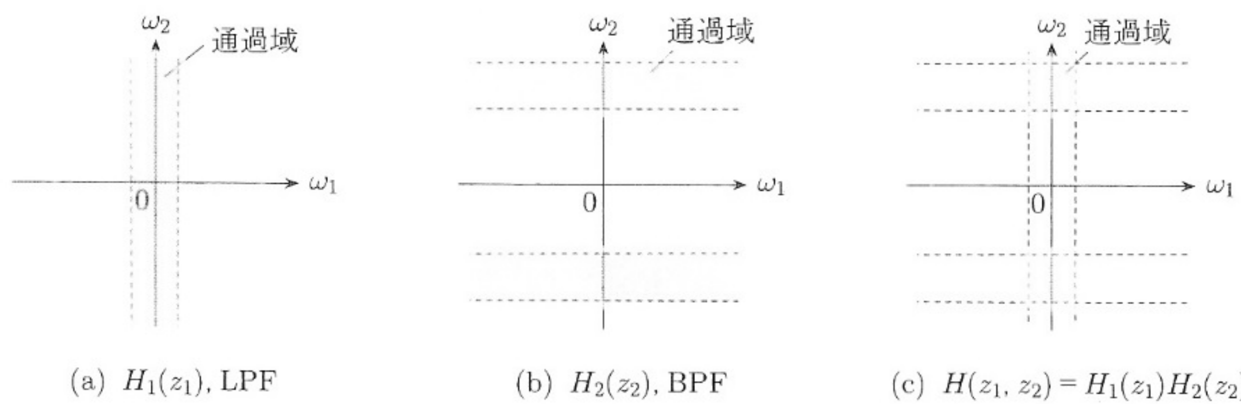

図 **3.19**　分離型フィルタの振幅特性

Separable Filter

- There are limits on realizable frequency characteristics.
- Which can be realizable by a separable filter?

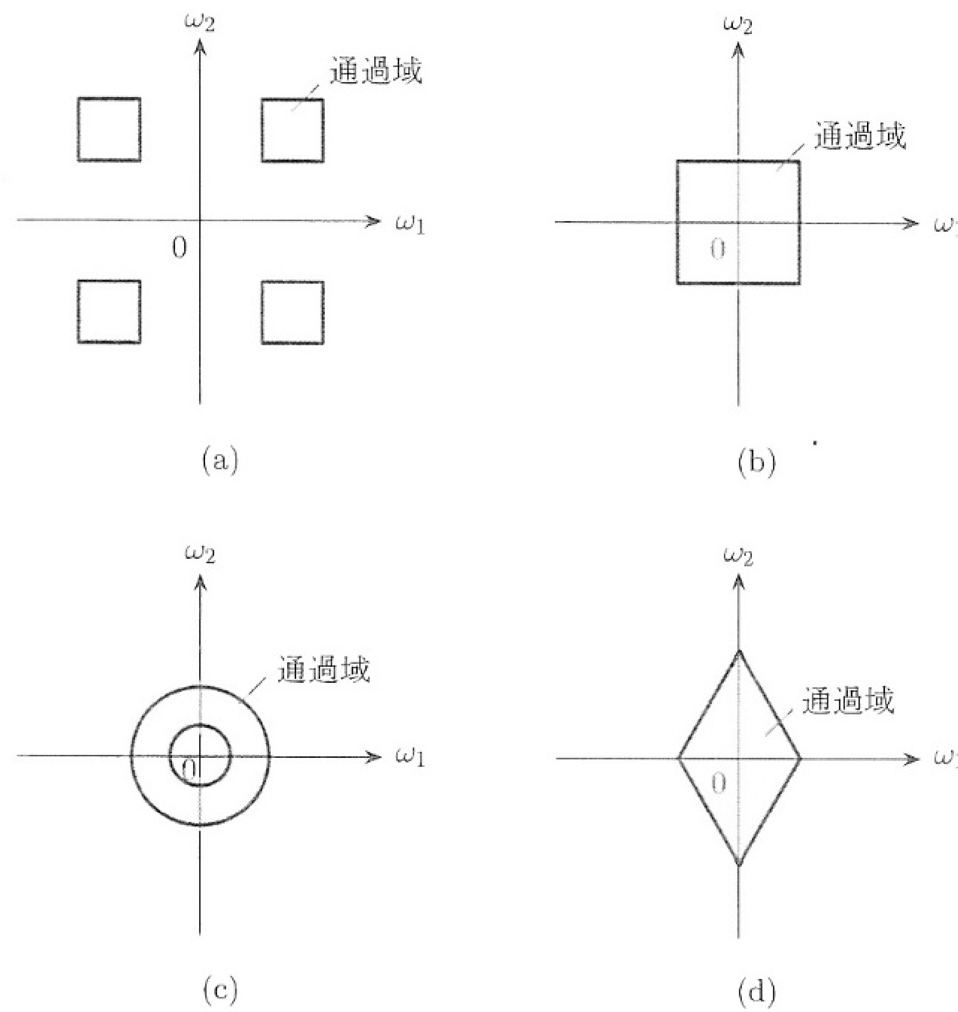

図 **3.20**　例題 3.9

Filter Processing in the Frequency Domain

- Can be perform by FFT.
- Manipulate image spectrum directly.

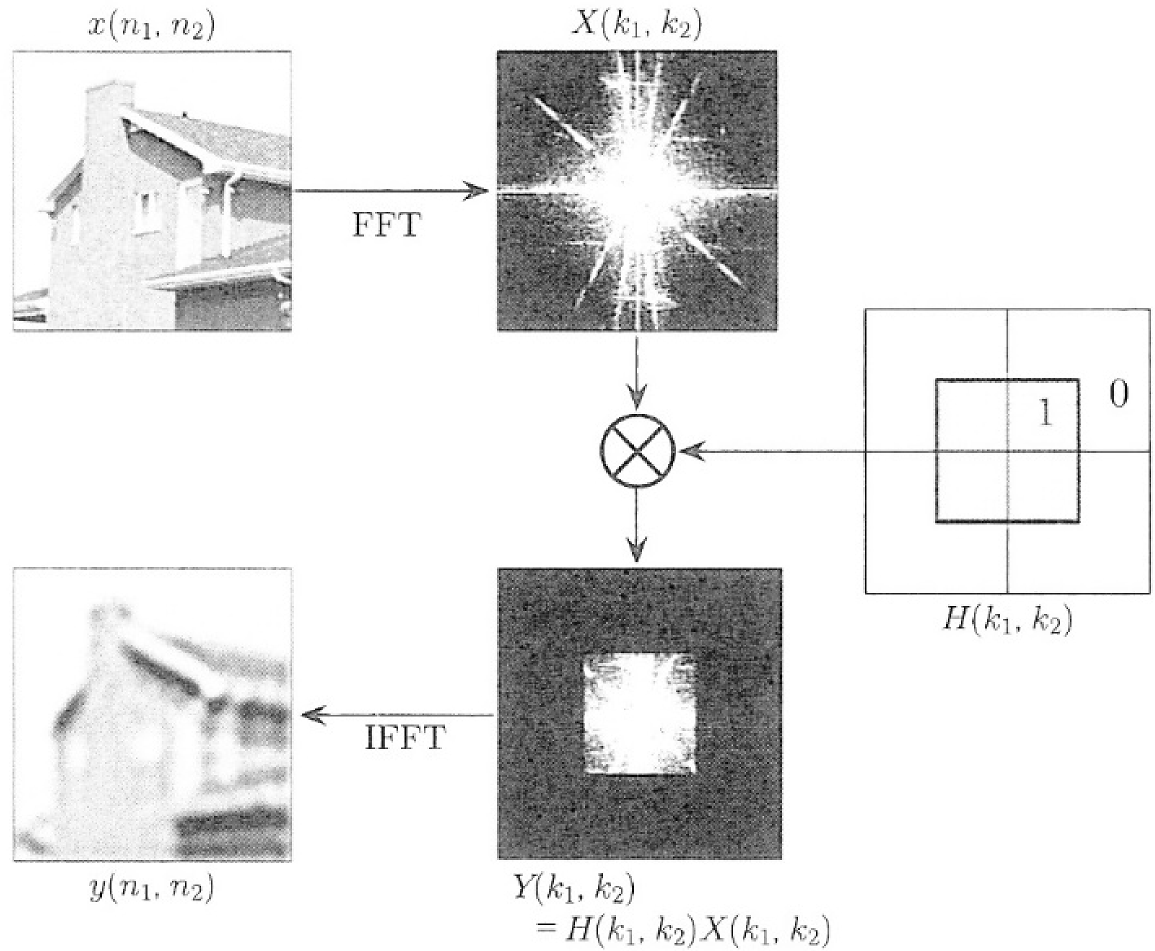

図 3.21 周波数領域でのフィルタ処理手順