Advanced Information Engineering

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Spectral Representation of Signal

- Signal representation by frequencies corresponds to frequency domain representation.
- Decompose signals into sinusoidal waves and deal with each frequency component.
- Treatment based on complex sinusoidal signals

$$g_a(x, y) = Ae^{j(\Omega_1 x + \Omega_2 y)}, \quad j = \sqrt{-1}$$

By Euler's formula, g_a is given by

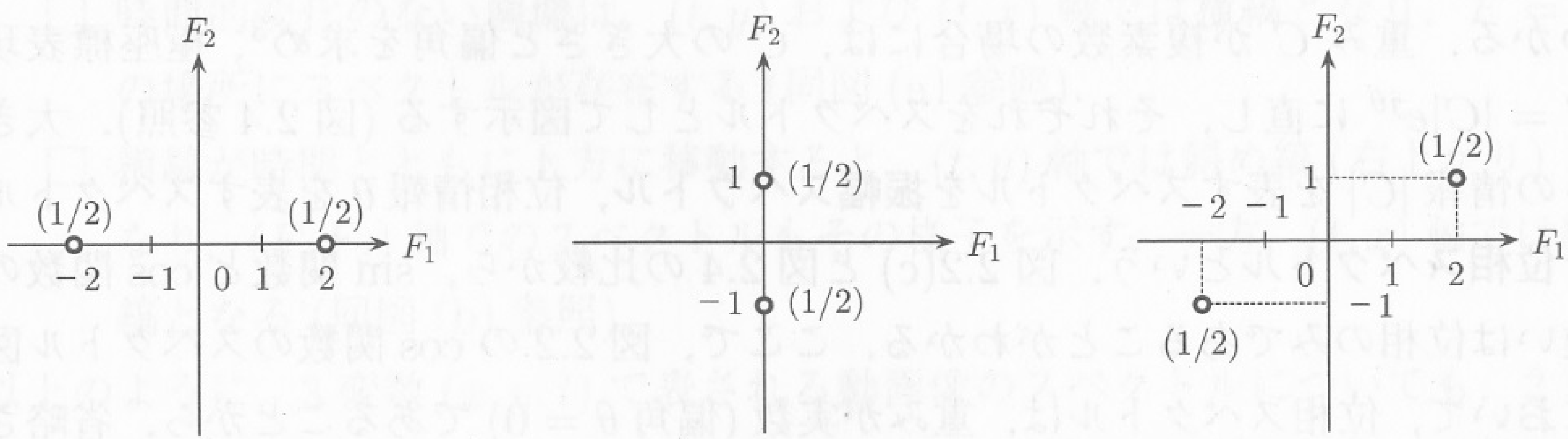
$$g_a(x, y) = \cos(\Omega_1 x + \Omega_2 y)$$

$$g_a(x, y) = (A/2)e^{j(\Omega_1 x + \Omega_2 y)} + (A/2)e^{-j(\Omega_1 x + \Omega_2 y)}$$

Spectral Representation of Signal

$$g_a(x,y) = (A/2)e^{j(\Omega_1 x + \Omega_2 y)} + (A/2)e^{-j(\Omega_1 x + \Omega_2 y)}$$

- The signal is represented by the sum of two complex sinusoidal waves with their weight A/2.
- Why is the spectral graph given by the figures below ?
- What is ○ in the spectral graphs?



(a) $F_1=2,\ F_2=0$ (b) $F_1=0,\ F_2=1$ (c) $F_1=2,\ F_2=1$

図 **2.2** $g_a(x,y)=\cos(2\pi F_1 x + 2\pi F_2 y)$ のスペクトル表現, $(\cdot)$ 内はスペクトル値を示す

Relationship between signal and spectrum

- What is the origin of frequency domain ?
- What will happen if the circle become farther away form the origin?
- What will happen about the direction of modulation ?

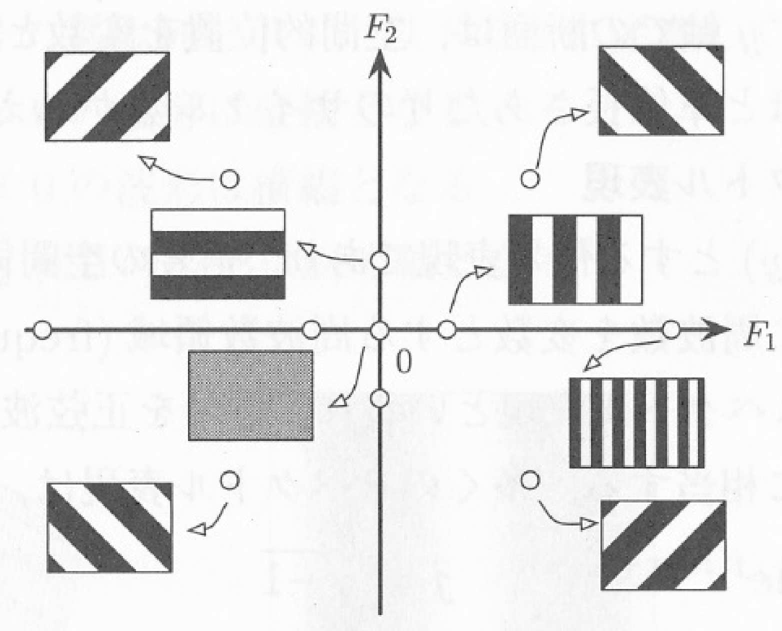


図 2.3 スペクトルと信号の関係

Example

$$g_a(x, y) = Ae^{j(\Omega_1 x + \Omega_2 y)}, \quad j = \sqrt{-1}$$

How is the above expression rewritten ?

$$g_a(x, y) = \sin(\Omega_1 x + \Omega_2 y)$$

↓

Answer

$$g_a(x, y) = Ae^{j(\Omega_1 x + \Omega_2 y)}, \quad j = \sqrt{-1}$$

How is the above expression rewritten ?

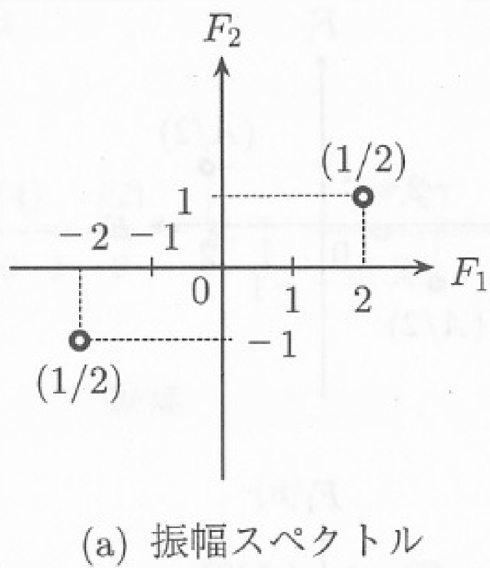
$$g_a(x, y) = \sin(\Omega_1 x + \Omega_2 y)$$

$$g_a(x, y) = (A/2j)e^{j(\Omega_1 x + \Omega_2 y)} + (-A/2j)e^{-j(\Omega_1 x + \Omega_2 y)}$$
$$= (A/2 \cdot e^{-j\pi/2})e^{j(\Omega_1 x + \Omega_2 y)} + (A/2 \cdot e^{j\pi/2})e^{-j(\Omega_1 x + \Omega_2 y)}$$

Amplitude and Phase Spectra

- When the weight is a complex number, calculate amplitude and phase angle and depict them as amplitude spectrum and phase spectrum.

$$g_a(x,y) = (A/2j)e^{j(\Omega_1 x+\Omega_2 y)} + (-A/2j)e^{-j(\Omega_1 x+\Omega_2 y)}$$
$$= (A/2\cdot e^{-j\pi/2})e^{j(\Omega_1 x+\Omega_2 y)} + (A/2\cdot e^{j\pi/2})e^{-j(\Omega_1 x+\Omega_2 y)}$$



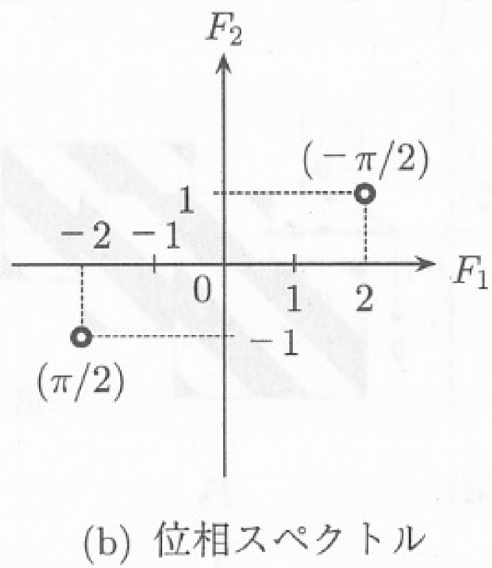


図 **2.4** $g_a(x,y) = \sin(2\pi F_1 x + 2\pi F_2 y), (F_1 = 2, F_2 = 1)$

Spectrum of Video Signals

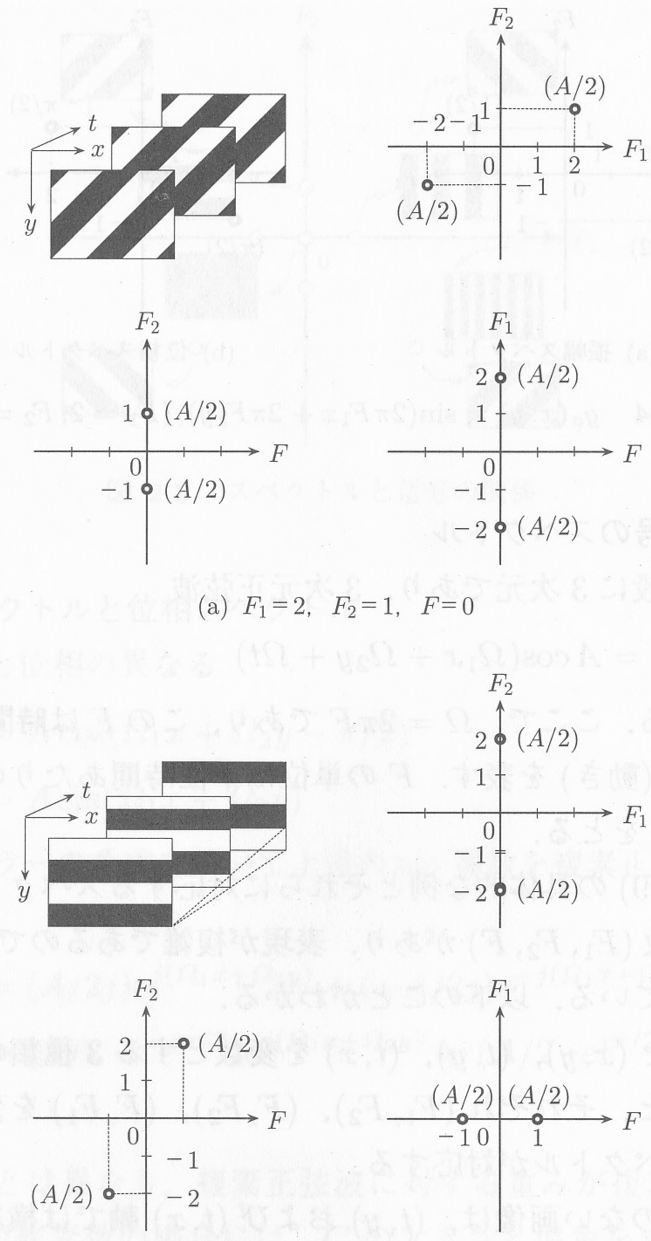
- Video signal is 3-dimensional and given by

$$g_a(x, y, t) = A\cos(\Omega_1 x + \Omega_2 y + \Omega t)$$

- Assume $\Omega=2\pi F$. What are F and Ω?

Specrtum of Video

- What is the type of signal (a) ?
- What is the type of signal (b) ?



(a) $F_1 = 2,\ F_2 = 1,\ F = 0$

(b) $F_1 = 0,\ F_2 = 2,\ F = 1$

図 **2.5** $g_a(x, y) = A\cos(2\pi(F_1 x + F_2 y + Ft))$

Exercise Example

- Specify 2D continuous signals ga(x,y) given by the spectral figures below.
- Illustrate black and white sketches of the above 2D signals.

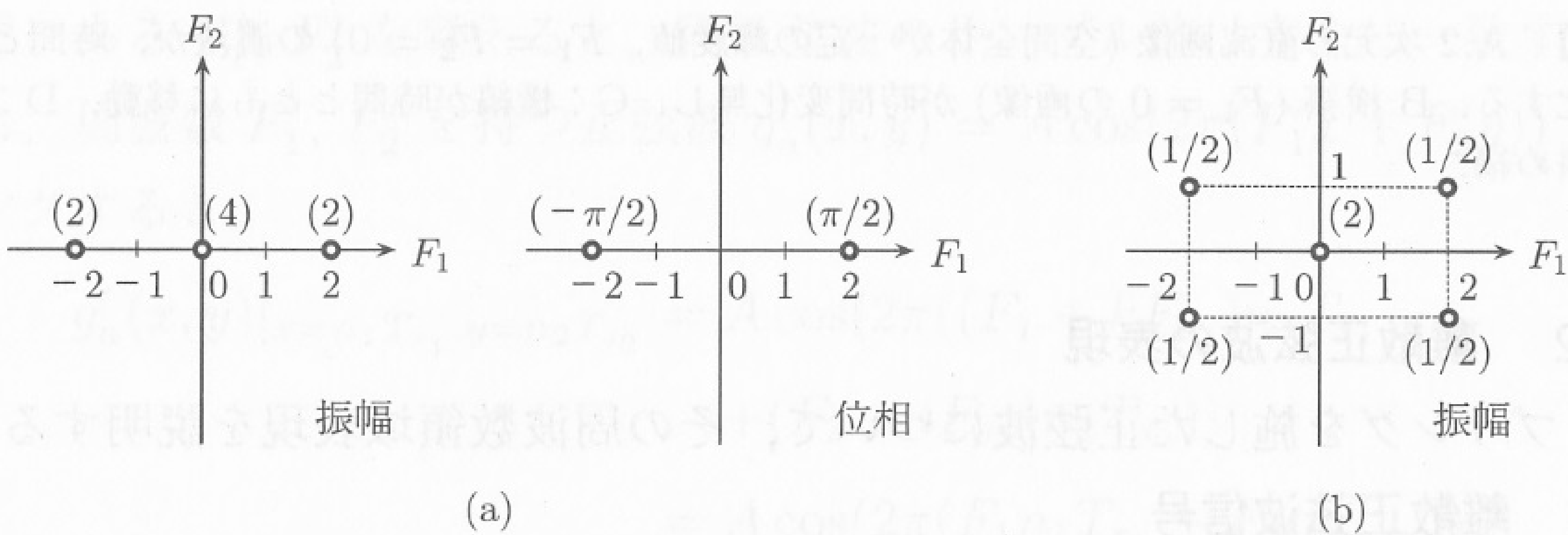


図 **2.6**　例題 2.1

Answers

- (a) $g_a(x,y)=4\times 1+2\times \exp(j\pi/2)\exp(j2\pi 2x)+2\times \exp(-j\pi/2)\exp(-j2\pi 2x)$

  $=4+2\times 2\times \sin(-4\pi x)$

  $=4-4\times \sin(4\pi x)$

  $=4+4\cos(4\pi x+\pi/2)$

- (b) $g_a(x,y)=2\times 1+0.5\times \exp(j2\pi(2x+y))+0.5\times \exp(-j2\pi(2x+y))+0.5\times \exp(j2\pi(2x-y))+0.5\times \exp(-j2\pi(2x-y))$

  $=2+\cos(2\pi(2x+y))+\cos(2\pi(2x-y))$

  $=2+2\cos(4\pi x)\cos(2\pi y)$

Answer

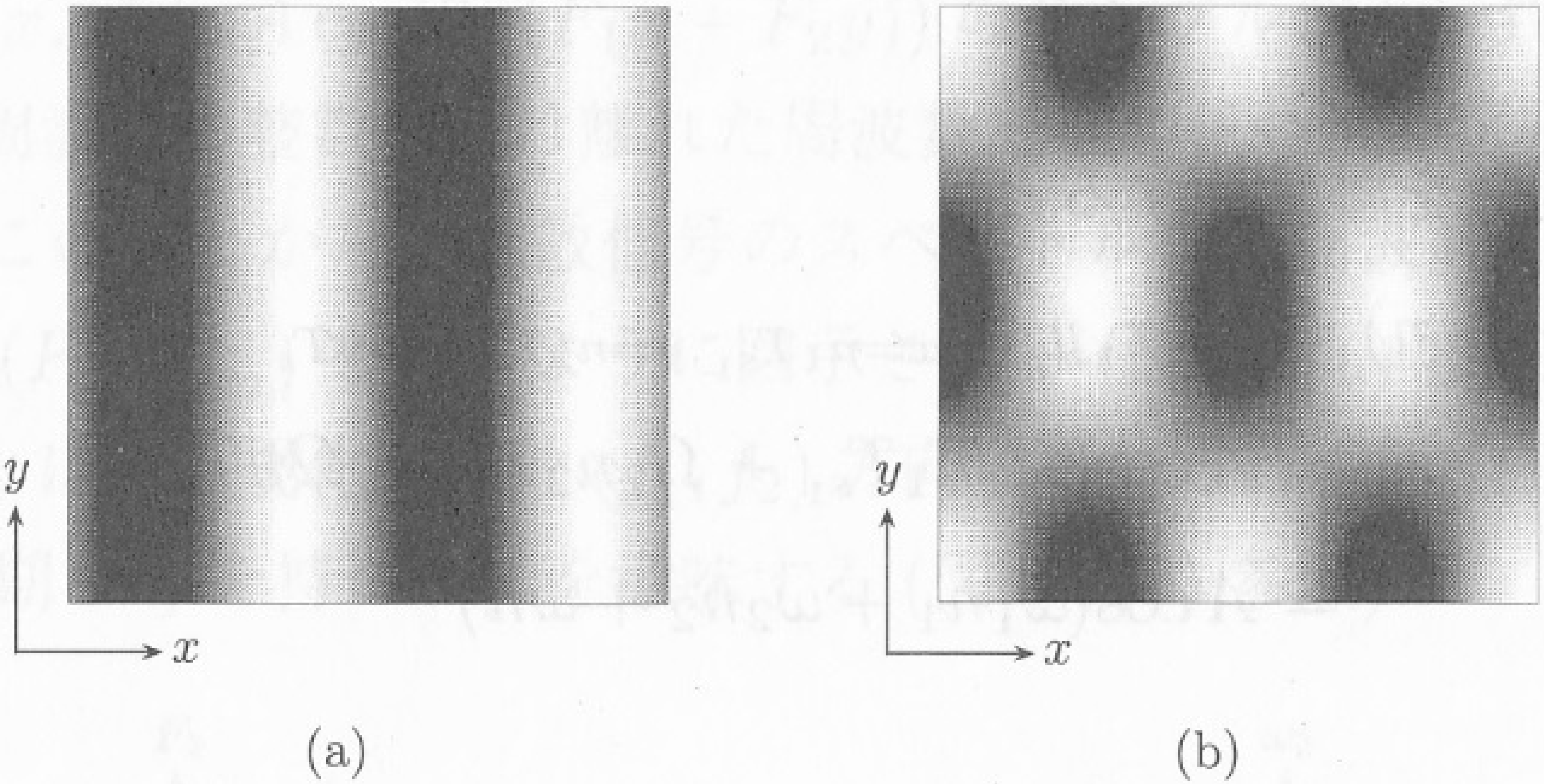


図 **2.7**　例題 2.1 の解答

Exercise Examples

- What kinds of images correspond to 3D spectra A, B, C, and D?

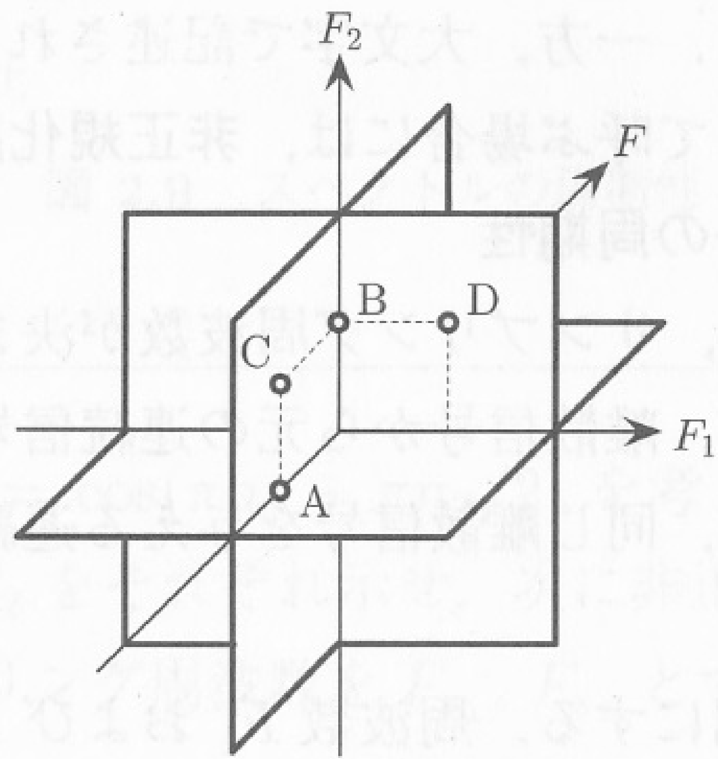


図 **2.8**　例題 2.2

Answers

- A : 2D direct current Image（constant intensity for all pixels, $F_1=F_2=0$）which changes the intensity temporally.
- B : still horizontal stripe（image with $F_1=0$）
- C : horizontal stripe（ $F_1=0$ の画像）moving vertically with time
- D : Slanted stripe unchanging with time

Discrete Sinusoidal Wave Signal

- Discretize $g_a(x,y,t)$ by sampling.

$$
\begin{aligned}
g(n_1, n_2, n) &= g_a(x, y, t)\big|_{x=n_1 T_{s_1}, y=n_2 T_{s_2}, t=nT_s} \\
&= A\cos(\Omega_1 n_1 T_{s_1} + \Omega_2 n_2 T_{s_2} + \Omega n T_s) \\
&= A\cos(\omega_1 n_1 + \omega_2 n_2 + \omega n)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1 &= 2\pi f_1 = \Omega_1 T_{s_1} = 2\pi F_1/F_{s_1} \\
\omega_2 &= 2\pi f_2 = \Omega_2 T_{s_2} = 2\pi F_2/F_{s_2} \\
\omega &= 2\pi f = \Omega T_s = 2\pi F/F_s
\end{aligned}
$$

f:normalized frequency, ω:normalized angular frequency

F:non-normalized frequency, Ω:non-normalized angular frequency

Periodicity of Frequency Spectrum

When

$$F'_1 = F_1 + kF_{s_1}$$

$$F'_2 = F_2 + iF_{s_2}$$

For

$$g_a(x, y) = A\cos(2\pi(F_1 x + F_2 y))$$

and

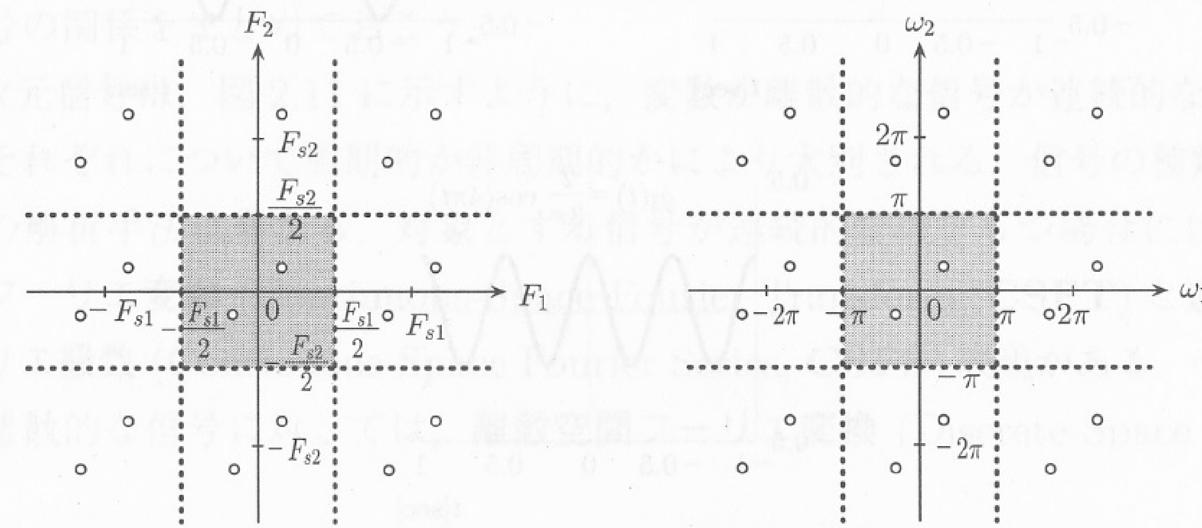
$$g'_a(x, y) = A\cos(2\pi(F'_1 x + F'_2 y))$$

their sample values are identical.*(proof skipped*)

Sinusoidal waves with frequencies different by the sampling frequency multiplied by integer cannot be distinguished.

Periodicity of Frequency Spectrum

- Sinusoidal waves with frequencies different by the sampling frequency multiplied by integer cannot be distinguished.
- What kind of signals in the figure?



(a) 非正規化　(b) 正規化

図 2.9　スペクトルの周期性

Spectrum of discrete signals

Exercise Example

- Let’s consider $g(n_1, n_2) = \cos(\pi n_1 + \pi n_2/2)$.
- Calculate normalized frequencies f_1 and f_2.
- When the sampling frequencies are F_{S1} and F_{S2}, Calculate non-normalized frequencies $F_1$and F_2.

Answers

- Since ω_1,ω_2 are π, $\pi/2$, respectively, $f_1=1/2, f_2=1/4$
- $F_1=F_{S1}/2, F_2=F_{S2}/4$

Fourier Analysis of Signal

- A non-sinusoidal wave is generated by adding two sinusoidal waves with different frequencies.
- There are cases where an non-sinusoidal wave is decomposed into plural sinusoidal waves.
- Since the law of superposition is satisfied for linear systems, the process for a non-sinusoidal wave comes down to those for plural sinusoidal waves by Fourier analysis.

Fourier Transform of Discrete Signals

- The Fourier transform of 1-dimensioncal non-periodic discrete signal is given

$$\mathrm{g(n)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(e^{j\omega})e^{j\omega n}d\omega$$

$$\mathrm{G}(\omega) = \sum_{n=-\infty}^{\infty} g(n)e^{-j\omega n}$$

Fourier Transform of Discrete Signals

- The Fourier transform of 2-dimensioncal non-periodic discrete signal is given by

$$g(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(e^{j\omega_1}, e^{j\omega_2})\, e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1\, d\omega_2$$

$$G(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

Amplitude and Phase Spectrums

Even $g(n_1, n_2)$ is a real valued function, $G(\omega_1, \omega_2)$ is generally a complex valued function.

As $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$, complex values are represented by polar coordinate system and draw spectrum by calculating amplitude and phase spectrums.

Exercise Example

- Perform Fourier transform the following 2-dimensional non-periodic discrete signal.
- Furthermore, as $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$ by representing by polar coordinate and calculate amplitude and phase spectrums.
- Draw the amplitude spectrum if possible.

$$g(n_1, n_2) = \begin{cases} 1 & (0 \leqq n_1 \leqq L_1 - 1, \text{ and } 0 \leqq n_2 \leqq L_2 - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Answer

$$G(\omega_1, \omega_2) = \sum_{n_1=0}^{L_1-1} \sum_{n_2=0}^{L_2-1} e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$= \sum_{n_1=0}^{L_1-1} e^{-j\omega_1 n_1} \cdot \sum_{n_2=0}^{L_2-1} e^{-j\omega_2 n_2}$$

$$= \frac{1 - e^{-j\omega_1 L_1}}{1 - e^{-j\omega_1}} \cdot \frac{1 - e^{-j\omega_2 L_2}}{1 - e^{-j\omega_2}}$$

Answer

$$= \frac{1 - e^{-j\omega_1 L_1}}{1 - e^{-j\omega_1}} \cdot \frac{1 - e^{-j\omega_2 L_2}}{1 - e^{-j\omega_2}}$$

$$= \frac{e^{\frac{-j\omega_1 L_1}{2}}(e^{\frac{j\omega_1 L_1}{2}} - e^{\frac{-j\omega_1 L_1}{2}})}{e^{\frac{-j\omega_1}{2}}(e^{\frac{j\omega_1}{2}} - e^{\frac{-j\omega_1}{2}})} \cdot \frac{e^{\frac{-j\omega_1 L_1}{2}}(e^{\frac{j\omega_1 L_1}{2}} - e^{\frac{-j\omega_1 L_1}{2}})}{e^{\frac{-j\omega_1}{2}}(e^{\frac{j\omega_1}{2}} - e^{\frac{-j\omega_1}{2}})}$$

$$= e^{\frac{-j\omega_1(L_1-1)}{2}} \cdot \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot e^{\frac{-j\omega_2(L_2-1)}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}$$

Answer

$$= e^{\frac{-j\omega_1(L_1-1)}{2}} \cdot \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot e^{\frac{-j\omega_2(L_2-1)}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}$$

$$= \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}} \cdot e^{j(\frac{-\omega_1(L_1-1)}{2}+\frac{-\omega_2(L_2-1)}{2})}$$

$$A(\omega_1,\omega_2) = \left|\frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}\right|$$ ⇦ 図示すると

$$\theta(\omega_1,\omega_2) = \left|\frac{-\omega_1(L_1-1)}{2} + \frac{-\omega_2(L_2-1)}{2}\right|$$

Answer

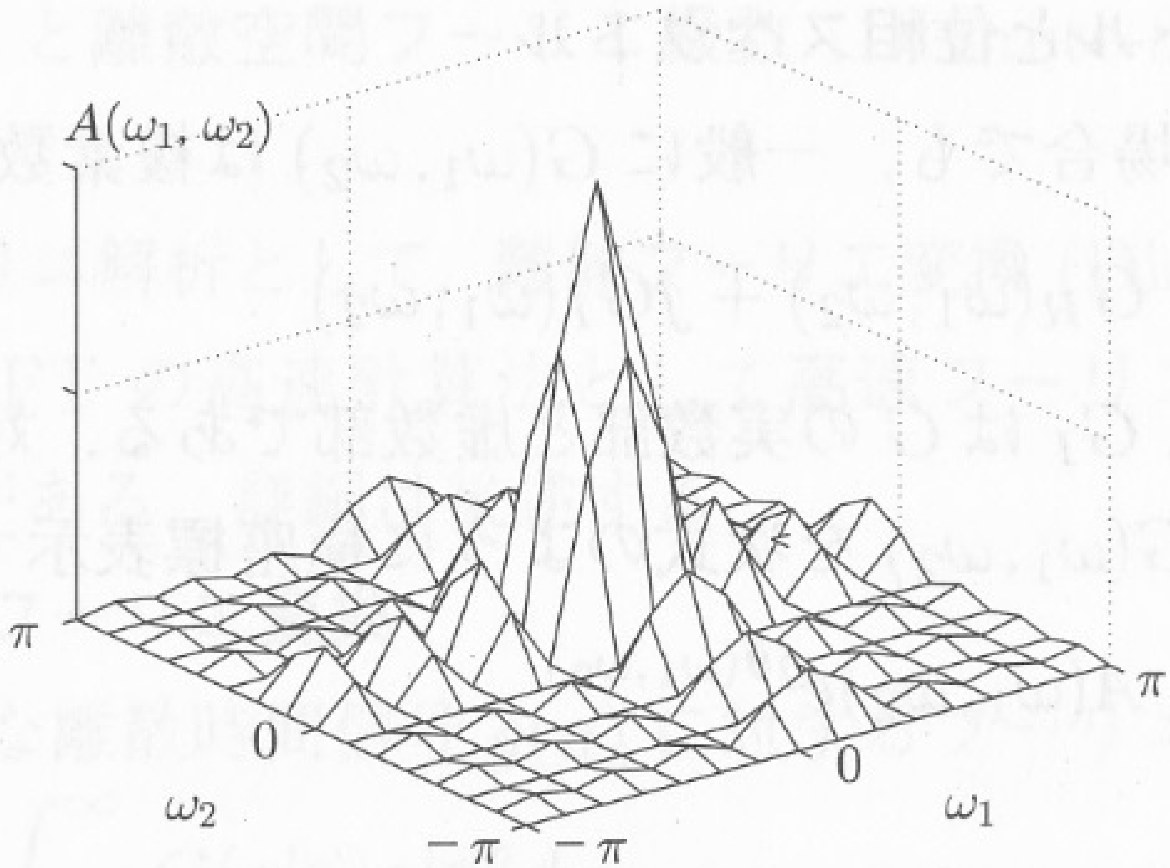
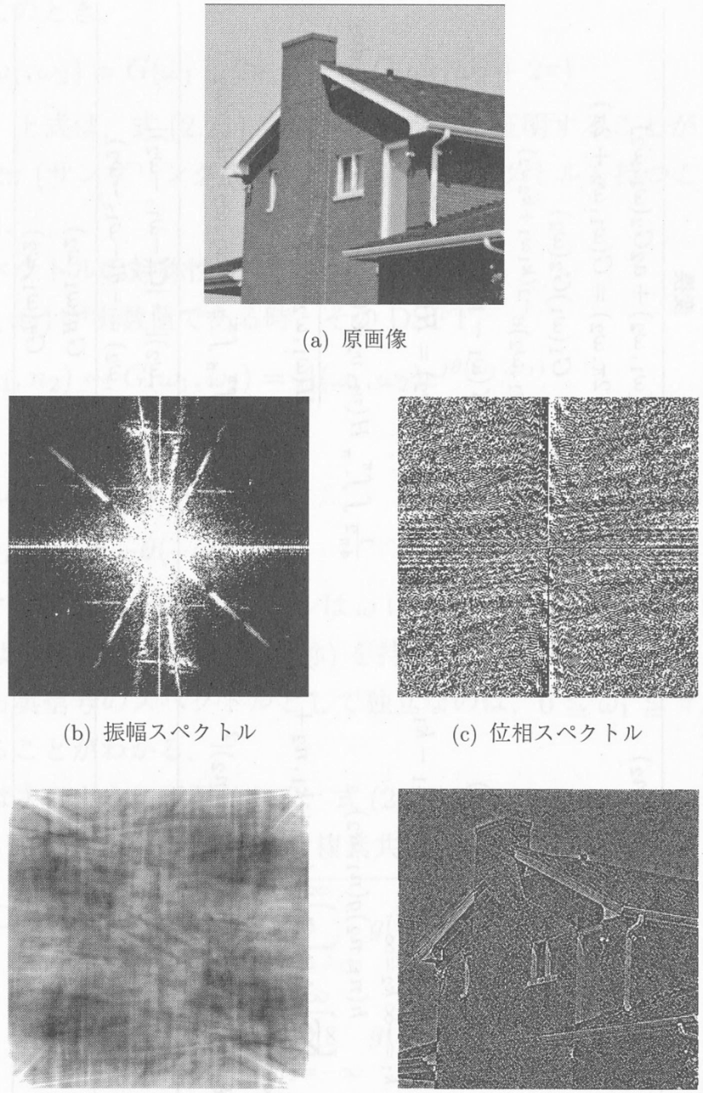


図 **2.12**　振幅スペクトル例 ($L_1 = L_2 = 8$)

Amplitude-only and Phase-only Images



(a) 原画像

(b) 振幅スペクトル

(c) 位相スペクトル

(d) 振幅限定画像

(e) 位相限定画像

図 **2.13**　振幅限定画像と位相限定画像

Symmetry of Spectrum

- In case where $\mathrm{g}(n_1, n_2)$ is a real valued function, its discrete Fourier transform given by

$G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$ satisfies

$$A(\omega_1, \omega_2) = A(-\omega_1, -\omega_2)$$

$$\theta(\omega_1, \omega_2) = -\theta(-\omega_1, -\omega_2)$$

（proof skipped）

Amplitude spectrum : even symmetry

Phase spectrum : odd symmetry

Signal Shift

- Signal $g(n_1, n_2)$ and its discrete Fourier transform $G(\omega_1, \omega_2)$
- For integers k_1 k_2, and signal $g(n_1 - k_1, n_2 - k_2)$, its discrete Fourier transform is given by $G(\omega_1, \omega_2)e^{-i(\omega_1 k_1 + \omega_2 k_2)}$. (proof skipped)
- No effect on phase spectrum
- Why ?

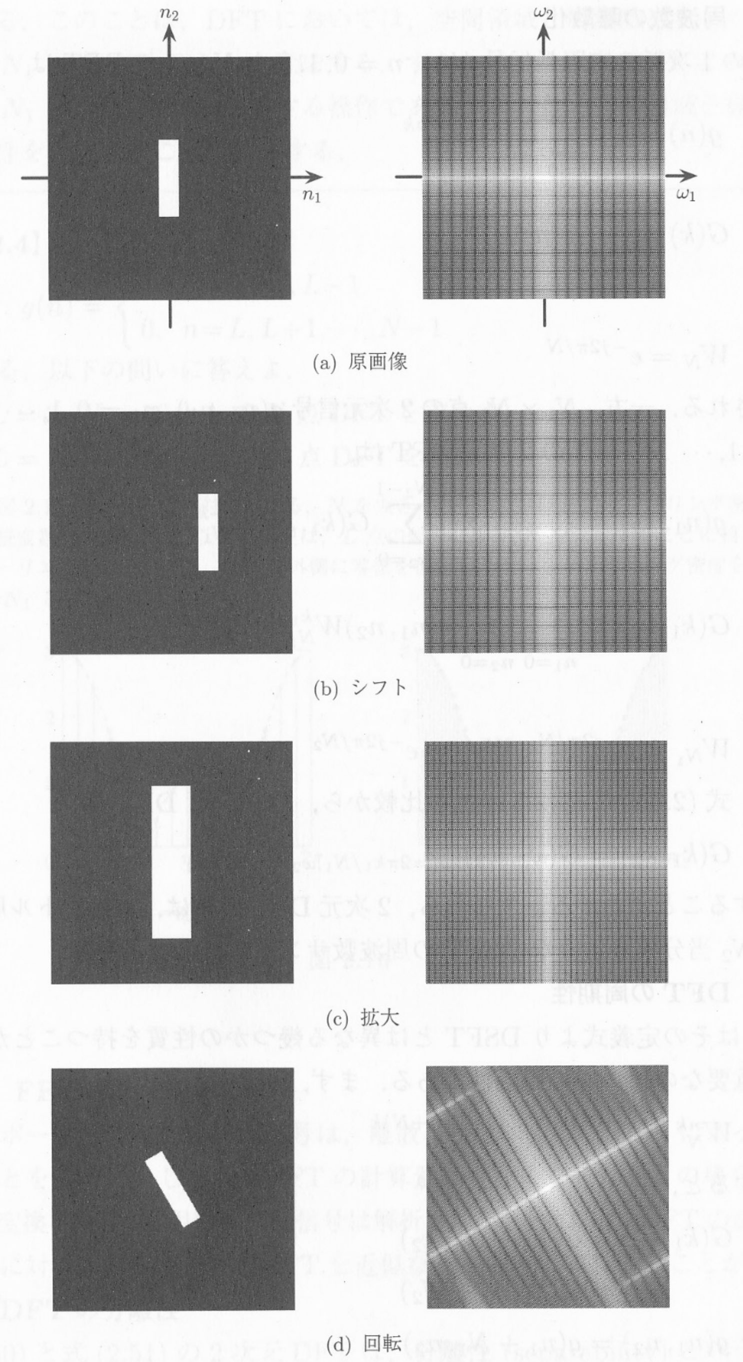


図 2.14 スペクトル計算例 (振幅)