Advanced Information Engineering

#6 November 9 (Mon), 2020

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Discrete Sinusoidal Wave Signal

- Discretize $g_a(x,y,t)$ by sampling.

$$
\begin{aligned}
g(n_1, n_2, n) &= g_a(x, y, t)\big|_{x=n_1 T_{s_1}, y=n_2 T_{s_2}, t=nT_s} \\
&= A\cos(\Omega_1 n_1 T_{s_1} + \Omega_2 n_2 T_{s_2} + \Omega n T_s) \\
&= A\cos(\omega_1 n_1 + \omega_2 n_2 + \omega n)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1 &= 2\pi f_1 = \Omega_1 T_{s_1} = 2\pi F_1 / F_{s_1} \\
\omega_2 &= 2\pi f_2 = \Omega_2 T_{s_2} = 2\pi F_2 / F_{s_2} \\
\omega &= 2\pi f = \Omega T_s = 2\pi F / F_s
\end{aligned}
$$

f:normalized frequency, ω:normalized angular frequency

F:non-normalized frequency, Ω:non-normalized angular frequency

Periodicity of Frequency Spectrum

When

$$F'_1 = F_1 + kF_{s_1}$$

$$F'_2 = F_2 + iF_{s_2}$$

For

$$g_a(x, y) = A\cos(2\pi(F_1 x + F_2 y))$$

nd

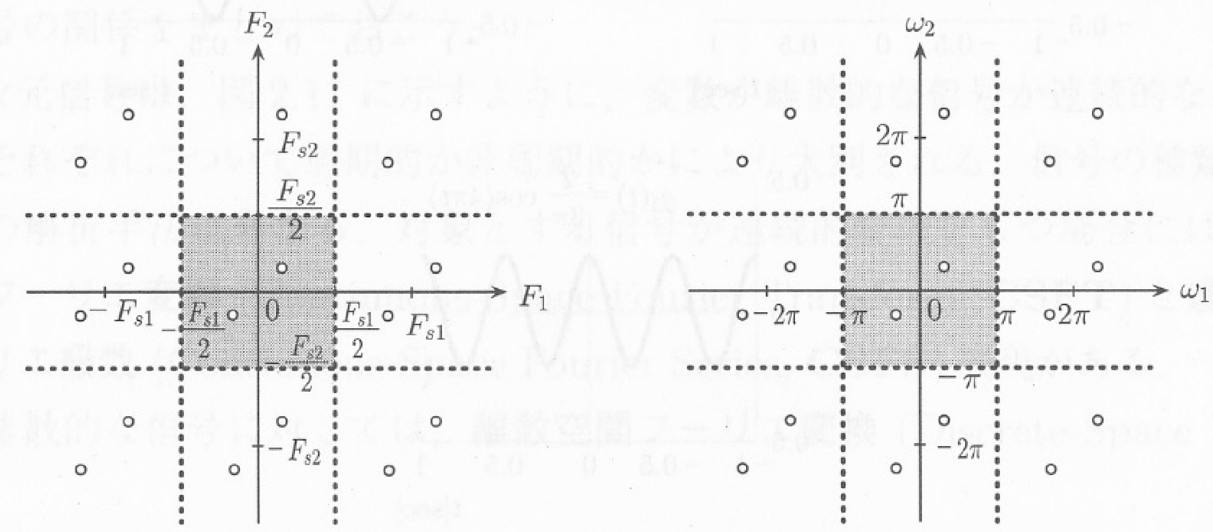
$$g'_a(x, y) = A\cos(2\pi(F'_1 x + F'_2 y))$$

their sample values are identical.*(proof skipped)*

Sinusoidal waves with frequencies different by the sampling frequency multiplied by integer cannot be distinguished.

Periodicity of Frequency Spectrum

- Sinusoidal waves with frequencies different by the sampling frequency multiplied by integer cannot be distinguished.
- What kind of signals in the figure?



(a) 非正規化　(b) 正規化

図 2.9　スペクトルの周期性

Spectrum of discrete signals

Exercise Example

- Let's consider $g(n_1, n_2) = \cos(\pi n_1 + \pi n_2/2)$.
- Calculate normalized frequencies f_1 and f_2.
- When the sampling frequencies are F_{S1} and F_{S2}, Calculate non-normalized frequencies F_1 and F_2.

Answers

- Since ω_1,ω_2 are π, $\pi/2$, respectively, $f_1=1/2, f_2=1/4$
- $F_1=F_{S1}/2, F_2=F_{S2}/4$

Fourier Analysis of Signal

- A non-sinusoidal wave is generated by adding two sinusoidal waves with different frequencies.
- There are cases where a non-sinusoidal wave is decomposed into plural sinusoidal waves.
- Since the law of superposition is satisfied for linear systems, the process for a non-sinusoidal wave comes down to those for plural sinusoidal waves by Fourier analysis.

Fourier Transform of Discrete Signals

- The Fourier transform of 1-dimensioncal non-periodic discrete signal is given

$$\mathrm{g(n)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(e^{j\omega})e^{j\omega n}d\omega$$

$$\mathrm{G}(\omega) = \sum_{n=-\infty}^{\infty} g(n)e^{-j\omega n}$$

Fourier Transform of Discrete Signals

- The Fourier transform of 2-dimensioncal non-periodic discrete signal is given by

$$\mathrm{g}(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(e^{j\omega_1}, e^{j\omega_2})\, e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1\, d\omega_2$$

$$G(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

Amplitude and Phase Spectrums

Even $\mathrm{g}(n_1, n_2)$ is a real valued function, $G(\omega_1, \omega_2)$ is generally a complex valued function.

As $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$, complex values are represented by polar coordinate system and draw spectrum by calculating amplitude and phase spectrums.

Exercise Example

- Perform Fourier transform the following 2-dimensional non-periodic discrete signal.
- Furthermore, as $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1,\omega_2)}$ by representing by polar coordinate and calculate amplitude and phase spectrums.
- Draw the amplitude spectrum if possible.

$$g(n_1,n_2)= \begin{cases} 1 \ (0 \leqq n_1 \leqq L_1-1, \text{ and } 0 \leqq n_2 \leqq L_2-1) \\ 0 \ (\text{otherwise}) \end{cases}$$

Answer

$$G(\omega_1, \omega_2) = \sum_{n_1=0}^{L_1-1} \sum_{n_2=0}^{L_2-1} e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$= \sum_{n_1=0}^{L_1-1} e^{-j\omega_1 n_1} \cdot \sum_{n_2=0}^{L_2-1} e^{-j\omega_2 n_2}$$

$$= \frac{1 - e^{-j\omega_1 L_1}}{1 - e^{-j\omega_1}} \cdot \frac{1 - e^{-j\omega_2 L_2}}{1 - e^{-j\omega_2}}$$

Answer

$$= \frac{1 - e^{-j\omega_1 L_1}}{1 - e^{-j\omega_1}} \cdot \frac{1 - e^{-j\omega_2 L_2}}{1 - e^{-j\omega_2}}$$

$$= \frac{e^{\frac{-j\omega_1 L_1}{2}}\left(e^{\frac{j\omega_1 L_1}{2}} - e^{\frac{-j\omega_1 L_1}{2}}\right)}{e^{\frac{-j\omega_1}{2}}\left(e^{\frac{j\omega_1}{2}} - e^{\frac{-j\omega_1}{2}}\right)} \cdot \frac{e^{\frac{-j\omega_1 L_1}{2}}\left(e^{\frac{j\omega_1 L_1}{2}} - e^{\frac{-j\omega_1 L_1}{2}}\right)}{e^{\frac{-j\omega_1}{2}}\left(e^{\frac{j\omega_1}{2}} - e^{\frac{-j\omega_1}{2}}\right)}$$

$$= e^{\frac{-j\omega_1(L_1-1)}{2}} \cdot \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot e^{\frac{-j\omega_2(L_2-1)}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}$$

Answer

$$= e^{\frac{-j\omega_1(L_1-1)}{2}} \cdot \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot e^{\frac{-j\omega_2(L_2-1)}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}$$

$$= \frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}} \cdot e^{j\left(\frac{-\omega_1(L_1-1)}{2}+\frac{-\omega_2(L_2-1)}{2}\right)}$$

$$A(\omega_1,\omega_2) = \left|\frac{\sin\frac{\omega_1 L_1}{2}}{\sin\frac{\omega_1}{2}} \cdot \frac{\sin\frac{\omega_2 L_2}{2}}{\sin\frac{\omega_2}{2}}\right|$$

⇦ 図示すると

$$\theta(\omega_1,\omega_2) = \left|\frac{-\omega_1(L_1-1)}{2} + \frac{-\omega_2(L_2-1)}{2}\right|$$

Answer

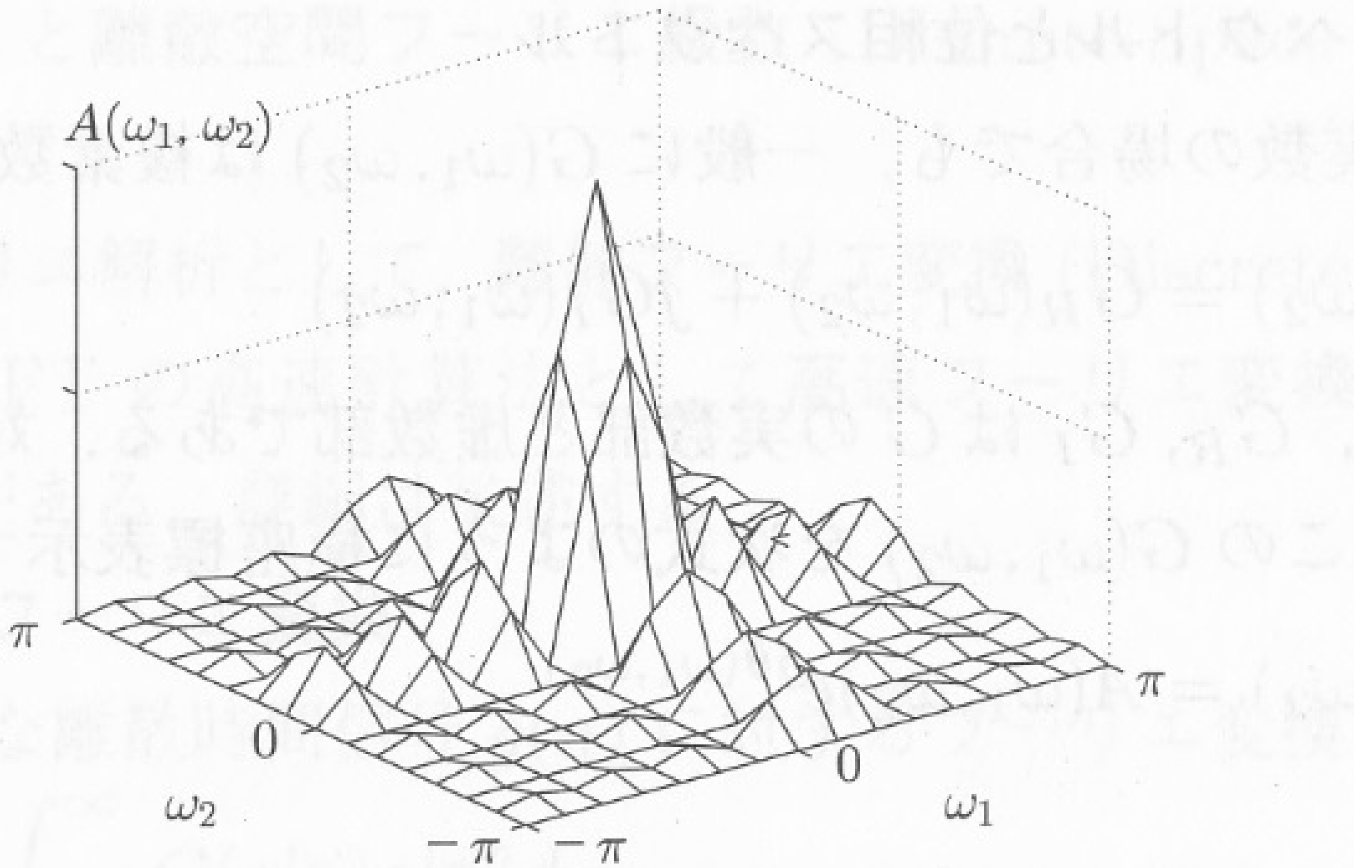
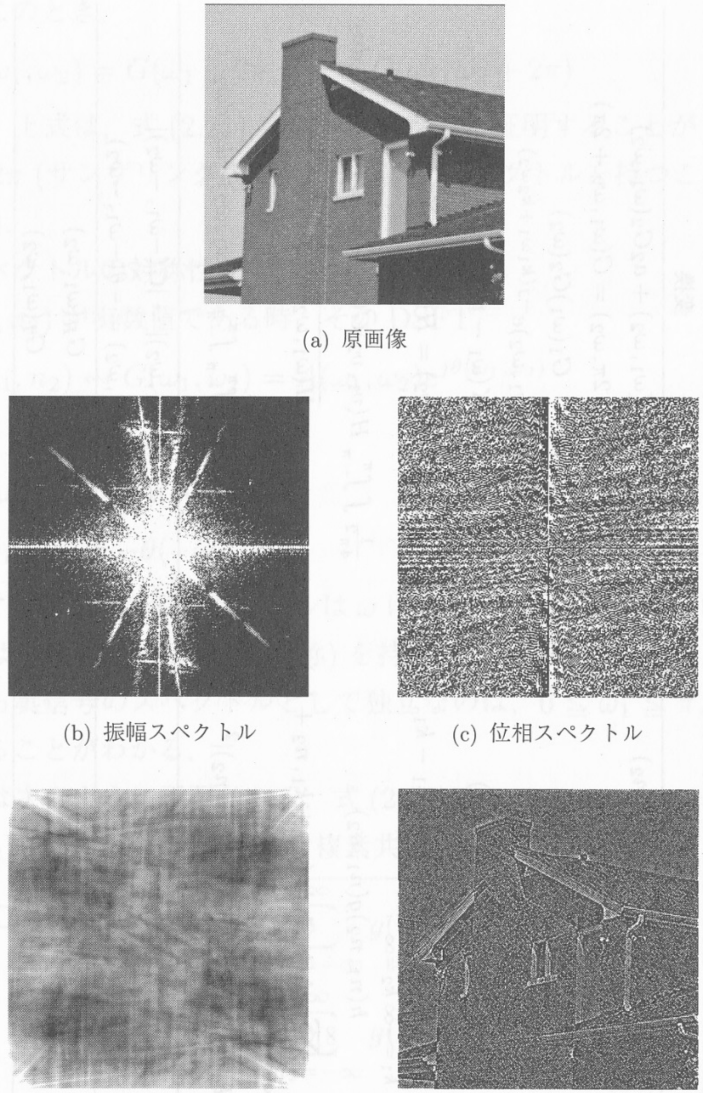


図 **2.12**　振幅スペクトル例 ($L_1 = L_2 = 8$)

Amplitude-only and Phase-only Images



(a) 原画像

(b) 振幅スペクトル

(c) 位相スペクトル

(d) 振幅限定画像

(e) 位相限定画像

図 **2.13**　振幅限定画像と位相限定画像

Symmetry of Spectrum

- In case where $\mathrm{g}(n_1, n_2)$ is a real valued function, its discrete Fourier transform given by

$G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1,\omega_2)}$ satisfies

$$A(\omega_1, \omega_2) = A(-\omega_1, -\omega_2)$$

$$\theta(\omega_1, \omega_2) = -\theta(-\omega_1, -\omega_2)$$

(proof skipped)

Amplitude spectrum : even symmetry

Phase spectrum : odd symmetry

Signal Shift

- Signal $g(n_1, n_2)$ and its discrete Fourier transform $G(\omega_1, \omega_2)$
- For integers k_1 k_2, and signal $g(n_1 - k_1, n_2 - k_2)$, its discrete Fourier transform is given by $G(\omega_1, \omega_2)e^{-i(\omega_1 k_1 + \omega_2 k_2)}$.
 (proof skipped)
- No effect on amplitude spectrum
- Why ?

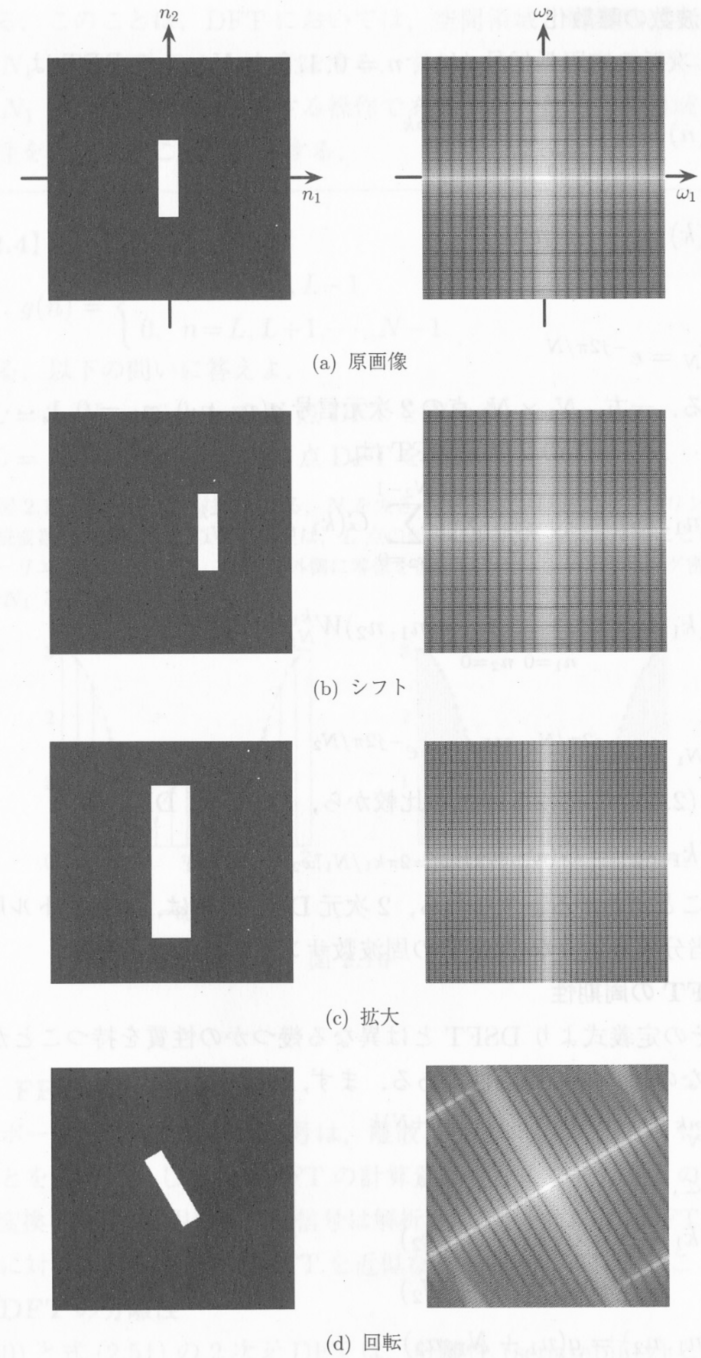


図 2.14 スペクトル計算例 (振幅)

Discrete Spatial Fourier Transform (DSFT)

$$\mathrm{g}(n_1, n_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(e^{j\omega_1}, e^{j\omega_2})\, e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1\, d\omega_2$$

$$G(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

The Fourier analysis performed by computer is done by FFT.

Discrete Spatial Fourier Transform (DSFT)

Consider the case where $g(n_1, n_2)$ is defined within finite domain $N_1 \times N_2$, i.e. 2-dimensial image signal.

$$G(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

Can the computer perform DSFT ?

Fast Fourier Transform（FFT)

- FFT is a fast calculation version of discrete Fourier transform（DFT）.

Discrete Fourier Transform（DFT）

- DSFT with frequency discretization
- In case where $g(n_1, n_2)$が is defined in $N_1 \times N_2$, a finite domain, i.d. 2-dimensional image.

$$g(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} G(k_1, k_2) W_{N_1}^{-k_1 n_1} W_{N_2}^{-k_2 n_2}$$

$$G(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) W_{N_1}^{k_1 n_1} W_{N_2}^{k_2 n_2}$$

Where

$$W_{N_1} = e^{-j2\pi/N_1}, W_{N_2} = e^{-j2\pi/N_2}$$

Thus the values of DFT are sampled ones of DSFT obtained by the intervals uniform intervals of spectrum period/N_1 and N_2.

Periodicity of DFT

$$W_N^{nk} = W_N^{n(k+N)} = W_N^{(n+N)k}$$

に注意すると,

$$G(k_1, k_2) = G(k_1 + N_1, k_2)$$

$$= G(k_1, k_2 + N_2)$$

$$g(n_1, n_2) = g(n_1 + N_1, n_2)$$

$$= g(n_1, n_2 + N_2)$$

- The number of independent points in both of the spatial and frequency domains is N1×N2 and we assume their periodicity and perform calculations.

Exercise Example

- 1 dimensional discrete signal of N points

$$g(n) = \begin{cases} 1, & n=0\cdots, L-1 \\ 0, & n=L, L+1,\cdots, N-1 \end{cases}$$

を考える．以下の問いに答えよ．

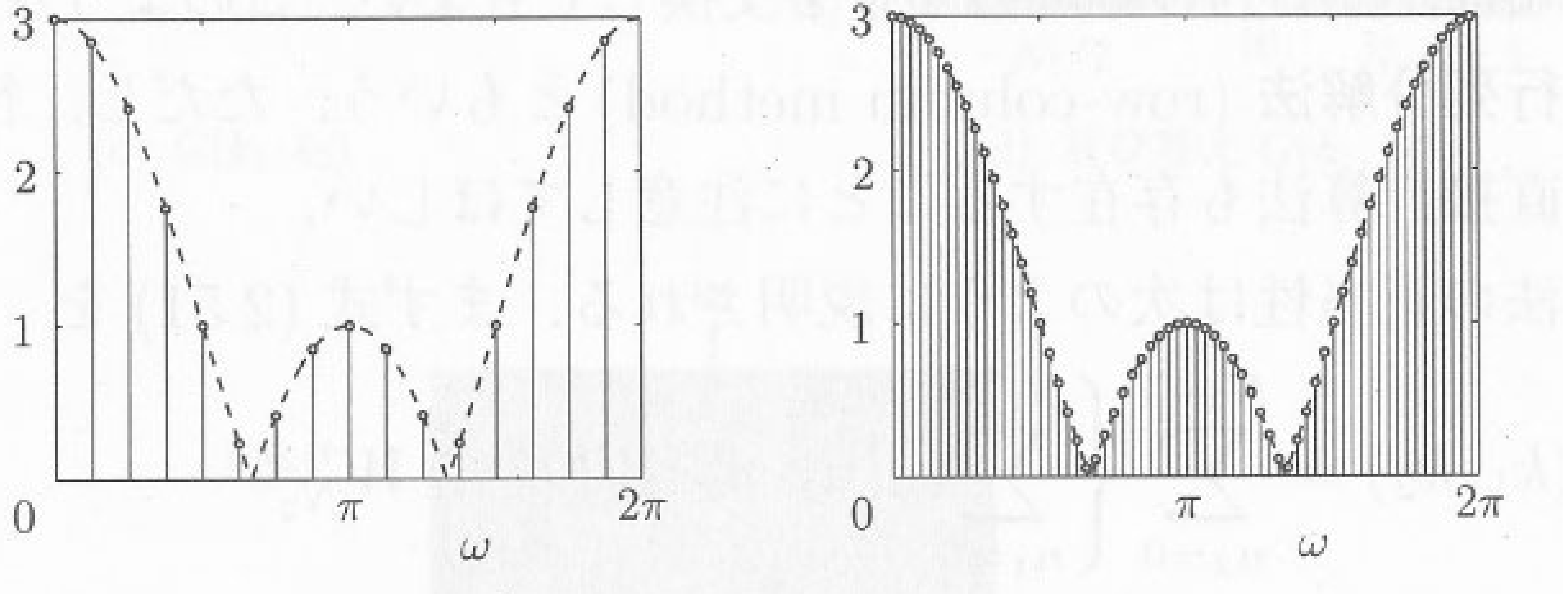
(a) $L = 3$, $N = 16$ として N 点 DFT を求めよ．

(b) $L = 3$, $N = 64$ として N 点 DFT を求めよ．

You can use a calculator to calculate amplitude.

Since (b) takes long time , please do (a).

Answer



(a) $L=3,\ N=16$ (b) $L=3,\ N=64$

- More larger N, more sufficient sampling density
- For Fourier image analysis, how to make sure to get sufficient sampling density?

Fast Fourier Transform（FFT）

- FFT is to make DFT（a lot of computational cost）faster.
- Without approximation error, it can perform DFT strictly.
- The method which takes advantage of Matrix decomposition method（decomposability of DFT）.

Matrix Decomposition (decomposability of DFT)

- Direct 2D DFT repeats N1×N2 DFT by N1×N2 times.
- In case of matrix decomposition, for horizontal row data perform N1 1D DFT by N2 times, then for column data perform N2 1D DFT by N1 times.

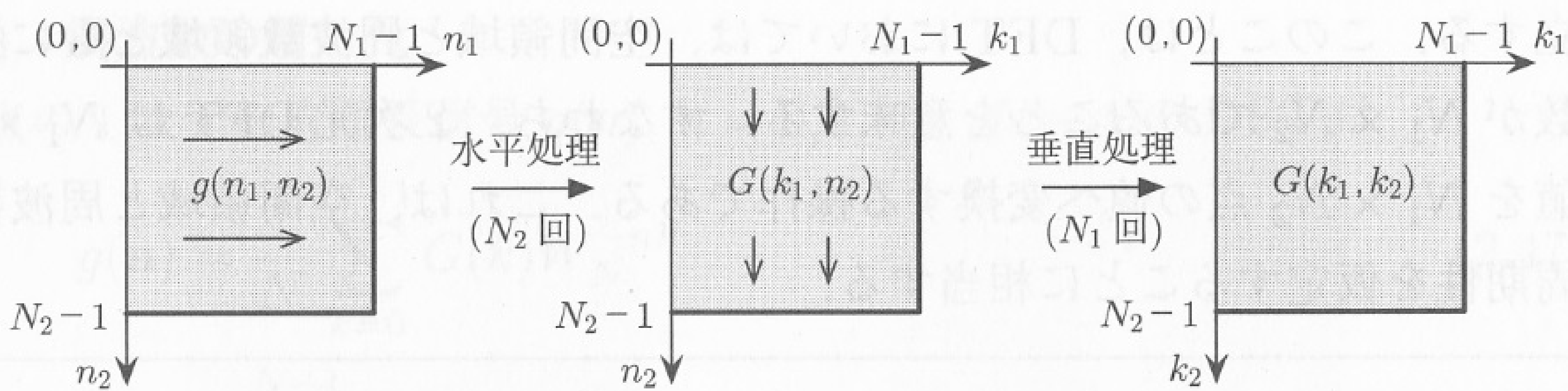


図 **2.16**　行列分解法

Example

- (a): 2D image signal
- (b): for (a), perform DFT horizontally
- (c): for (b), perform DFT vertically.
- (d): By using the periodicity of DFT, put DC component at the center.

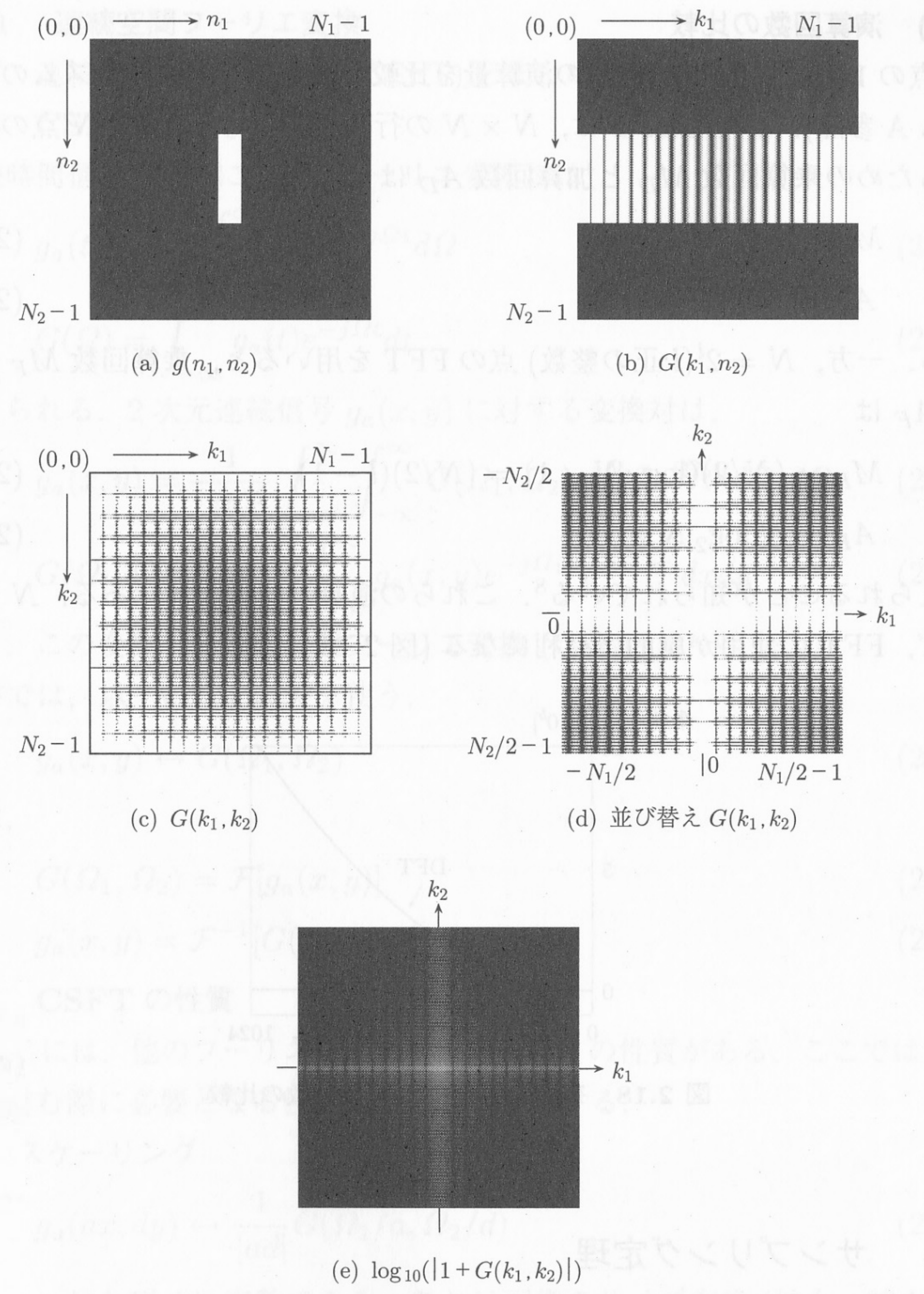
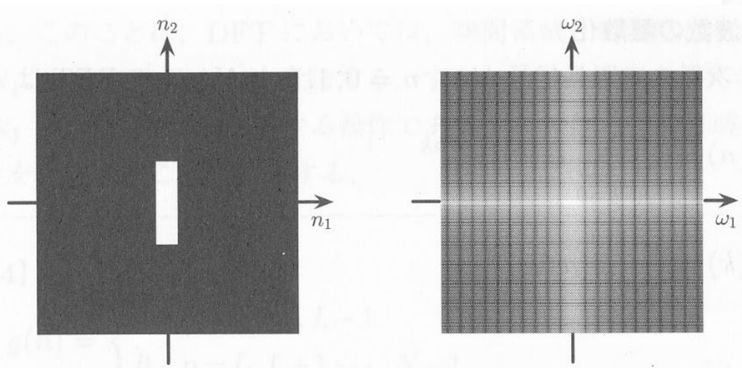


図 2.17 行列分解法による FFT 計算例



Comparison of Operation Number

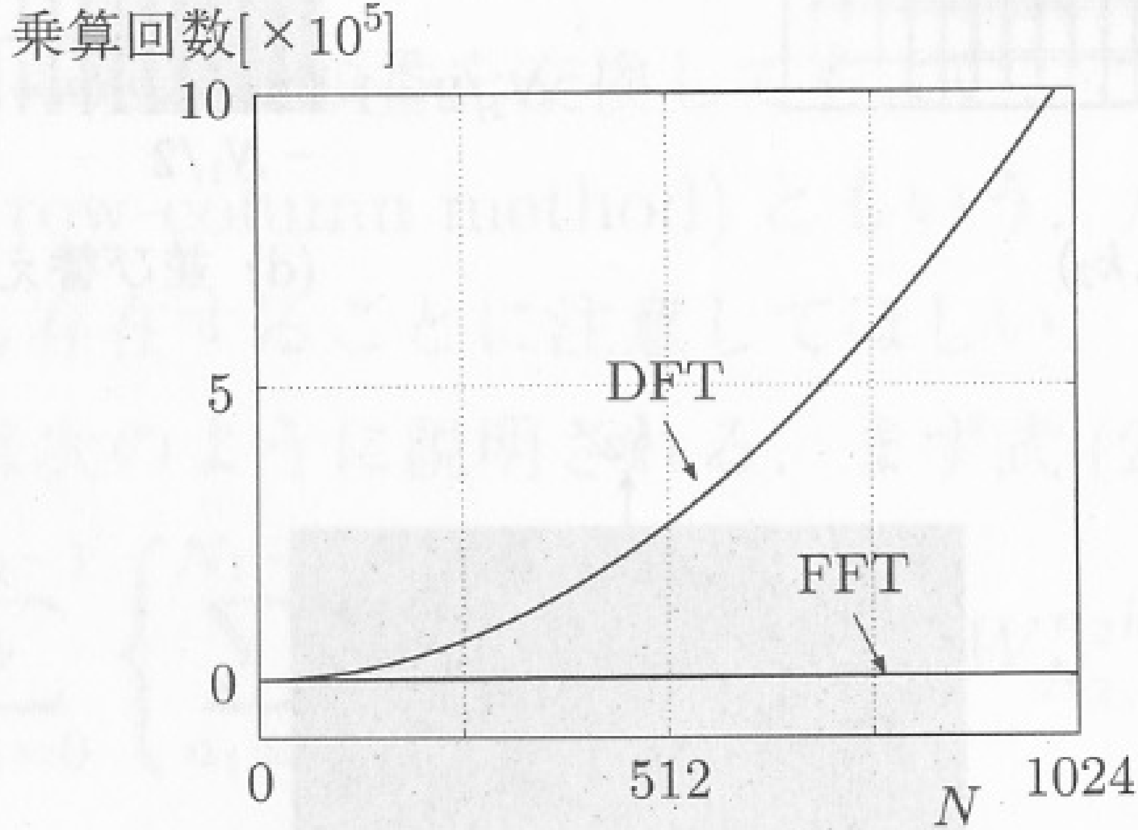


図 2.18　DFT と FFT の乗算回数の比較

Sampling Effect

- Sampling generally gives signals distortions (aliasing).

Sampling Theorem

- Theorem that give some condition to avoid the effects of sampling

Sampling Theorem

- Fig.(a): 2D continuous signal's bandwidth is limited by angular frequency Ωm1 and Ωm2. (No signal exists outside of the limited bandwidth.)
- Fig.(b): Assume rectangular sampling, 2D discrete signal has rectangular periodic spectrum.

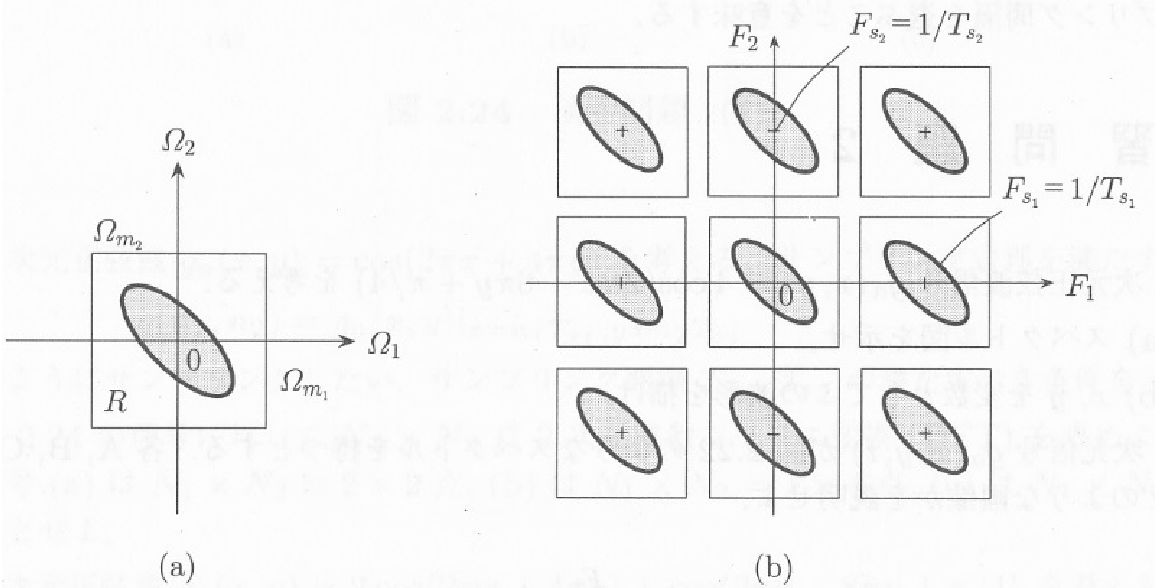


図 2.20　サンプリングの影響

Sampling Theorem

$$F_{s_1} = 1/T_{s_1} > 2F_{m_1}, \text{かつ } F_{s_2} = 1/T_{s_2} > 2F_{m_2}$$

No overlap exists for spectrum.

- Theoretically it is possible to reconstruct perfectly the original signal from sample values by filtering.

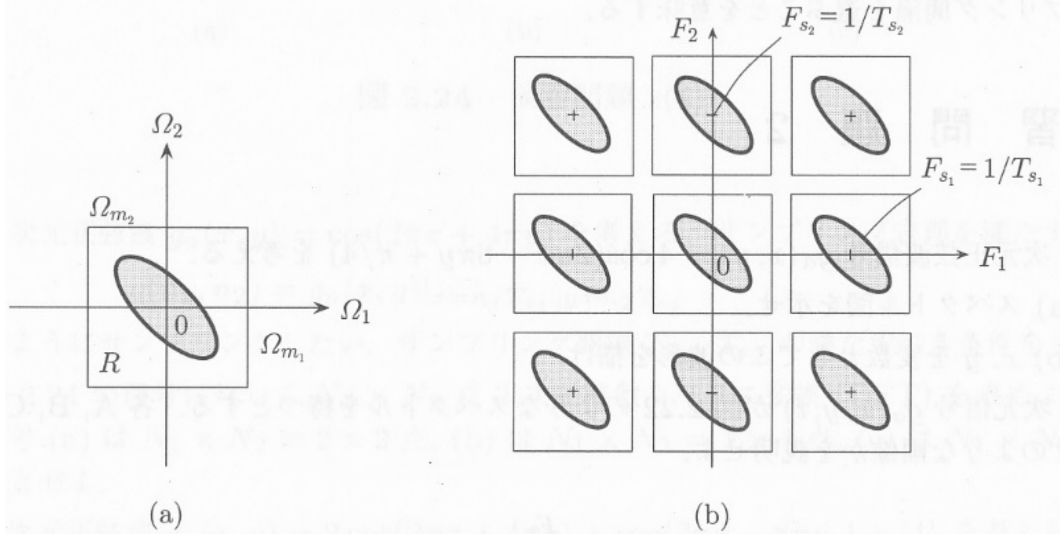
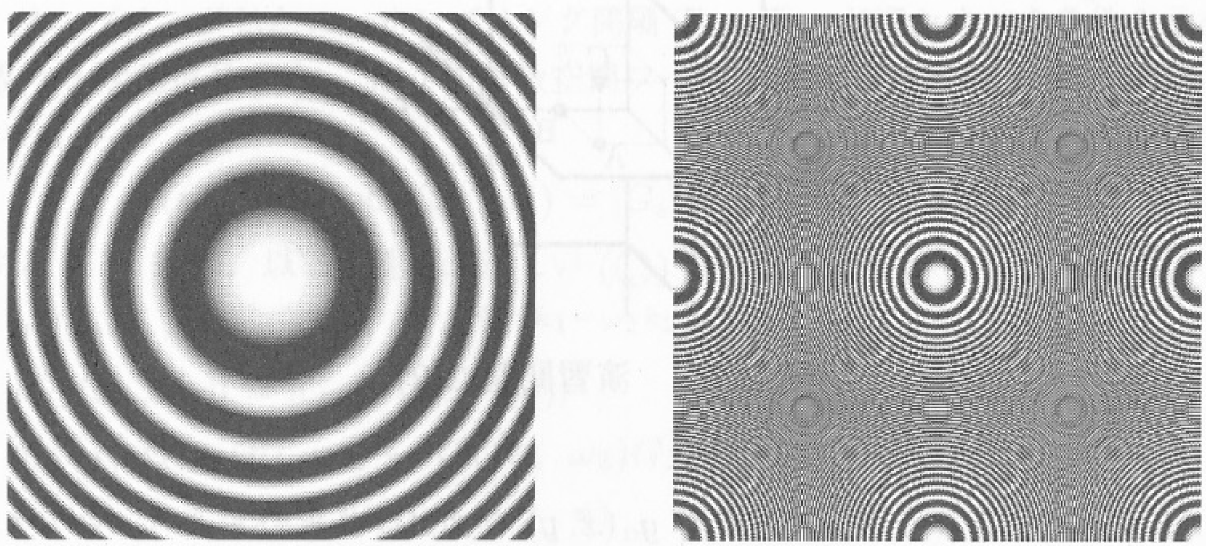


図 2.20 サンプリングの影響

折り返し歪み（Aliasing）

- By sampling without keeping sampling theorem, the spectrums overlap and distort the continuous signal. This distortion (overlap of spectrums) is called aliasing.



(a) エイリアジングなし　　(b) エイリアジングあり

図 **2.21**　エイリアジングの発生例

Example

$$g_a\ (x, y) = \cos(2\pi x + 4\pi y)$$

Calculate the maximum sapling intervals T_{s1} and T_{s2} to keep the sampling theorem.

Answer

- Since the spatial frequency is 1, 2 respectively in the x and y directions, the minimum sampling frequencies are 2, and 4 and their corresponding sampling intervals T_{s1} and T_{s2} are ½ and ¼, respectively.

Assignment #2

- Hand out Assignment #2