Advanced Information Engineering

#7 November 16 (Mon), 2020 Kenjiro T. Miura

Fourier Analysis of Signal

- A non-sinusoidal wave is generated by adding two sinusoidal waves with different frequencies.
- There are cases where an non-sinusoidal wave is decomposed into plural sinusoidal waves.
- Since the law of superposition is satisfied for linear systems, the process for a nonsinusoidal wave comes down to those for plural sinusoidal waves by Fourier analysis.

Fourier Transform of Discrete Signals

• The Fourier transform of 1-dimensioncal nonperiodic discrete signal is given

$$g(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(e^{j\omega}) e^{j\omega n} d\omega$$

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n}$$

Fourier Transform of Discrete Signals

 The Fourier transform of 2-dimensioncal nonperiodic discrete signal is given by

$$g(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(e^{j\omega_1}, e^{j\omega_2}) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$



Amplitude and Phase Spectrums

Even $g(n_1, n_2)$ is a real valued function, $G(\omega_1, \omega_2)$ is generally a complex valued function.

As $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$, complex values are represented by polar coordinate system and draw spectrum by calculating amplitude and phase spectrums.

Exercise Example

- Perform Fourier transform the following 2dimensional non-periodic discrete signal.
- Furthermore, as $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$ by representing by polar coordinate and calculate amplitude and phase spectrums.
- Draw the amplitude spectrum if possible.

$$g(n_1,n_2) = \begin{cases} 1 \ (0 \le n_1 \le L_1 - 1, \text{ and } 0 \le n_2 \le L_2 - 1) \\ 0 \ (\text{otherwise}) \end{cases}$$

$$G(\omega_1, \omega_2) = \sum_{n_1=0}^{L_1-1} \sum_{n_2=0}^{L_2-1} e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$=\sum_{n_1=0}^{L_1-1} e^{-j\omega_1 n_1} \cdot \sum_{n_2=0}^{L_2-1} e^{-j\omega_2 n_2}$$

$$=\frac{1-e^{-j\omega_{1}L_{1}}}{1-e^{-j\omega_{1}}}\cdot\frac{1-e^{-j\omega_{2}L_{2}}}{1-e^{-j\omega_{2}}}$$

$$=\frac{1-e^{-j\omega_{1}L_{1}}}{1-e^{-j\omega_{1}}}\cdot\frac{1-e^{-j\omega_{2}L_{2}}}{1-e^{-j\omega_{2}}}$$

$$=\frac{e^{\frac{-j\omega_{1}L_{1}}{2}}(e^{\frac{j\omega_{1}L_{1}}{2}}-e^{\frac{-j\omega_{1}L_{1}}{2}})}{e^{\frac{-j\omega_{1}}{2}}(e^{\frac{j\omega_{1}}{2}}-e^{\frac{-j\omega_{1}}{2}})}\cdot\frac{e^{\frac{-j\omega_{1}L_{1}}{2}}(e^{\frac{j\omega_{1}L_{1}}{2}}-e^{\frac{-j\omega_{1}L_{1}}{2}})}{e^{\frac{-j\omega_{1}}{2}}(e^{\frac{j\omega_{1}}{2}}-e^{\frac{-j\omega_{1}}{2}})}$$

$$=e^{\frac{-j\omega_1(L_1-1)}{2}}\cdot\frac{\sin\frac{\omega_1L_1}{2}}{\sin\frac{\omega_1}{2}}\cdot e^{\frac{-j\omega_2(L_2-1)}{2}}\cdot\frac{\sin\frac{\omega_2L_2}{2}}{\sin\frac{\omega_2}{2}}$$





$$A(\omega_1, \omega_2) = \left| \frac{\sin \frac{\omega_1 L_1}{2}}{\sin \frac{\omega_1}{2}} \cdot \frac{\sin \frac{\omega_2 L_2}{2}}{\sin \frac{\omega_2}{2}} \right| \leftarrow 义示する$$

$$\theta(\omega_1, \omega_2) = \left| \frac{-\omega_1(L_1 - 1)}{2} + \frac{-\omega_2(L_2 - 1)}{2} \right|$$



図 2.12 振幅スペクトル例 $(L_1 = L_2 = 8)$

Amplitude-only and Phase-only Images



(a) 原画像



Symmetry of Spectrum

• In case where $g(n_1, n_2)$ is a real valued function, its discrete Fourier transform given by $G(\omega_1, \omega_2) = A(\omega_1, \omega_2)e^{j\theta(\omega_1, \omega_2)}$ satisfies

$$A(\omega_1, \omega_2) = A(-\omega_1, -\omega_2)$$

$$\theta(\omega_1, \omega_2) = -\theta(-\omega_1, -\omega_2)$$

(proof skipped)

Amplitude spectrum : even symmetry Phase spectrum : odd symmetry

Signal Shift

- Signal $g(n_1, n_2)$ and its discrete Fourier transform $G(\omega_1, \omega_2)$
- For integers $k_1 k_2$, and signal $g(n_1 k_1, n_2 k_2)$, its discrete Fourier transform is given by

 $G(\omega_1, \omega_2)e^{-i(\omega_1k_1+\omega_2k_2)}.$ (proof skipped)

- No effect on phase spectrum
- Why?



図 2.14 スペクトル計算例

Discrete Spatial Fourier Transform (DSFT)

$$g(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(e^{j\omega_1}, e^{j\omega_2}) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

$$G(\omega_1, \omega_2) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

The Fourier analysis performed by computer is done by FFT.

Discrete Spatial Fourier Transform (DSFT)

Consider the case where $g(n_1, n_2)$ is defined within finite domain N₁×N₂, i.e. 2-dimensial image signal.

$$G(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

Can the computer perform DSFT?

Fast Fourier Transform (FFT)

• FFT is a fast calculation version of discrete Fourier transform (DFT) .

Discrete Fourier Transform (DFT)

- DSFT with frequency discretization
- In case where $g(n_1, n_2)$ is defined in N₁×N₂, a finite domain, i.d. 2-dimensional image.

$$g(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} G(k_1, k_2) W_{N_1}^{-k_1 n_1} W_{N_2}^{-k_2 n_2}$$
$$G(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) W_{N_1}^{k_1 n_1} W_{N_2}^{k_2 n_2}$$

Where

$$W_{N_1} = e^{-j2\pi/N_1}, W_{N_2} = e^{-j2\pi/N_2}$$

Thus the values of DFT are sampled ones of DSFT obtained by the intervals uniform intervals of spectrum period/N₁ and N₂.

Periodicity of DFT

$$W_N^{nk} = W_N^{n(k+N)} = W_N^{(n+N)k}$$

に注意すると、
 $G(k_1, k_2) = G(k_1 + N_1, k_2)$
 $= G(k_1, k_2 + N_2)$
 $g(n_1, n_2) = g(n_1 + N_1, n_2)$
 $= g(n_1, n_2 + N_2)$

 The number of independent points in both of the spatial and frequency domains is N1×N2 and we assume their periodicity and perform calculations.

Exercise Example

• 1 dimensional discrete signal of N points

 $g(n) = \begin{cases} 1, n=0\cdots, L-1\\ 0, n=L, L+1, \cdots, N-1 \end{cases}$ を考える.以下の問いに答えよ. (a) L = 3, N = 16 として N 点 DFT を求めよ. (b) L = 3, N = 64 として N 点 DFT を求めよ.

You can use a calculator to calculate amplitude. Since (b) takes long time , please do (a).



- Larger N, more sufficient sampling density
- For Fourier image analysis, how to make sure to get sufficient sampling density?

Fast Fourier Transform (FFT)

- FFT is to make DFT (a lot of computational cost) faster.
- Without approximation error, it can perform DFT strictly.
- The method which takes advantage of Matrix decomposition method (decomposability of DFT).

Matrix Decomposition (decomposability of DFT)

- Direct 2D DFT repeats N1×N2 DFT by N1×N2 times.
- In case of matrix decomposition, for horizontal row data perform N1 1D DFT by N2 times, then for column data perform N2 1D DFT by N1 times.



Example

- (a): 2D image signal
- (b): for (a), perform
 DFT horizontally
- (c): for (b), perform DFT vertically.
- (d): By using the periodicity of DFT, put DC component at the center.





(e) log₁₀(|1+*G*(*k*₁, *k*₂)|) 図 **2.17** 行列分解法による FFT 計算例

Comparison of Operation Number



図 2.18 DFT と FFT の乗算回数の比較

Sampling Effect

• Sampling generally gives signals distortions (aliasing).

Sampling Theorem

 Theorem that give some condition to avoid the effects of sampling

Sampling Theorem

- Fig.(a): 2D continuous signal's bandwidth is limited by angular frequency $\Omega m1$ and $\Omega m2$. (No signal exists outside of the limited bandwidth.)
- Fig.(b): Assume rectangular sampling, 2D discrete signal has rectangular periodic spectrum.



Sampling Theorem

 $F_{s_1} = 1/T_{s_1} > 2F_{m_1}, \text{tr} \supset F_{s_2} = 1/T_{s_2} > 2F_{m_2}$

No overlap exists for spectrum.

 Theoretically it is possible to reconstruct perfectly the original signal from sample values by filtering.



折り返し歪み (Aliasing)

 By sampling without keeping sampling theorem, the spectrums overlap and distort the continuous signal. This distortion (overlap of spectrums) is called aliasing.



Example

$$g_a(x,y) = \cos(2\pi x + 4\pi y)$$

Calculate the maximum sapling intervals T_{s1} and T_{s2} to keep the sampling theorem.

• Since the spatial frequency is 1, 2 respectively in the x and y directions, the minimum sampling frequencies are 2, and 4 and their corresponding sampling intervals T_{s1} and T_{s2} are $\frac{1}{2}$ and $\frac{1}{4}$, respectively.