Advanced Information Engineering

#8 November 30 (Mon), 2020

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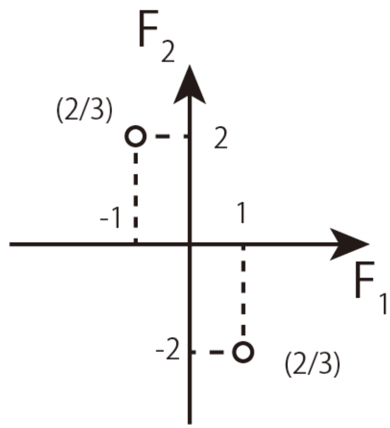
Assignment #2 Ex.1 Answer

Consider 2-dimensional sinusoidal wave signal

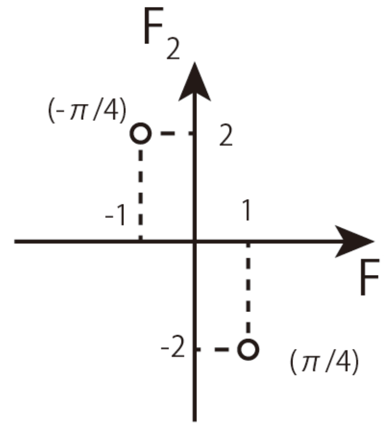
$g_a(x,y) = 3\cos(2\pi x - 4\pi y + \pi/4)$

Draw spectrum.

$g_a(x,y) = \frac{3}{2}\left(e^{j(2\pi(x-2y)+\frac{\pi}{4})} + e^{j(-2\pi(x-2y)-\frac{\pi}{4})}\right)$



Amplitude

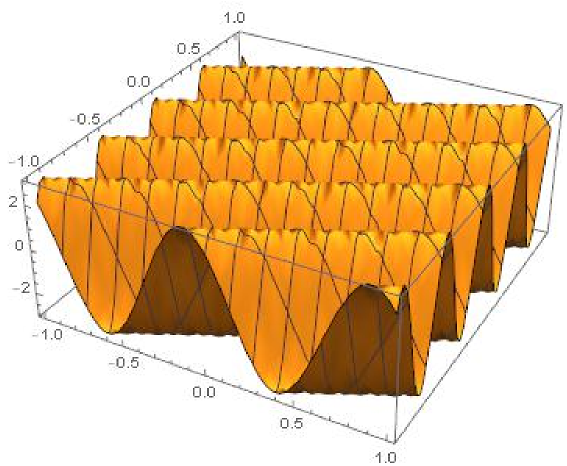


Phase

Assignment #2 Ex.1 Answer

Consider 2-dimensional sinusoidal wave signal
$g_a(x,y)=3 \cos (2 \pi x - 4 \pi y + \pi/4)$

Illustrate this wave assuming that its variables are x, y.



Assignment #2 Ex.2 Answer

Calculate a continuous signal which has spectrums given by Fig. 1.

(a) $g_a(x,y) = 2\cos(2\pi y) + \cos(4\pi y)$

(b) $g_a(x,y) = \cos(4\pi x + 2\pi y - \frac{\pi}{4})$

Assignment #2 Ex.3 Answer

Perform discrete spatial Fourier transform of each of the signals shown in Fig. 2.

(a) $G(\varpi_1,\varpi_2)$=1+$e^{-j\varpi_1}+e^{-j\varpi_2}$+$e^{-j(\varpi_1+\varpi_2)}$

$=4\cos(\frac{\varpi_1}{2})\cos(\frac{\varpi_2}{2})e^{-\frac{j(\varpi_1+\varpi_2)}{2}}$

(b) $G(\varpi_1,\varpi_2)$=1+$e^{-j\varpi_2}+e^{-2j\varpi_2}$+$e^{-3j\varpi_2}$

$=2\,(\cos(\frac{\varpi_2}{2})\cos(\frac{3\varpi_2}{2}))e^{-\frac{j3\varpi_2}{2}}$

(c) $G(\varpi_1,\varpi_2)$=$e^{j\varpi_1}+1+e^{-j\varpi_1}=1+2\cos(\varpi_1)$

Assignment #2 Ex.4 Answer

When a real-valued 2-dimensional discrete signal $g(n_1,n_2)$ satisfies that $g(n_1,n_2)=g(-n_1,-n_2)$, show that its discrete spatial Fourier transform (DSFT) $G(w_1,w_2)$ is real-valued.

$$G(\varpi_1,\varpi_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(\varpi_1,\varpi_2)e^{-j\varpi_1 n_1}\, e^{-j\varpi_2 n_2}$$

Assignment #2 Ex.4 Answer

$$
\begin{aligned}
G(\omega_1, \omega_2) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} g(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \\
&= g(0,0) + \sum_{n_1=1}^{\infty} g(n_1, 0)(e^{j\omega_1 n_1} + e^{-j\omega_1 n_1}) + \sum_{n_2=1}^{\infty} g(0, n_2)(e^{j\omega_2 n_2} + e^{-j\omega_2 n_2}) \\
&\quad + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (g(n_1, n_2)(e^{j(\omega_1 n_1 + \omega_2 n_2} + e^{-j(\omega_1 n_1 + \omega_2 n_2}) \\
&+ g(-n_1, n_2)(e^{j(\omega_1 n_1 + \omega_2 n_2} + e^{-j(\omega_1 n_1 + \omega_2 n_2}) \\
&= g(0,0) + \sum_{n_1=1}^{\infty} 2g(n_1, 0)\cos(\omega_1 n_1) + \sum_{n_2=1}^{\infty} 2g(0, n_2)\cos(\omega_2 n_2) \\
&\quad + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} 2(g(n_1, n_2) + g(-n_1, n_2))\cos(\omega_1 n_1 + \omega_2 n_2)
\end{aligned}
$$

Assignment #2 Ex.5 Answer

Consider a 2-dimensional sinusoidal wave $g_a(x,y)=\cos(2\pi x + 4\pi y)$. In order to satisfy the sampling theorem, we would like to sample as follows:

$g(n_1,n_2)=g_a(x,y)|x=n_1T_{S1},y=n_2 T_{S2}$.

Indicate the conditions for sampling intervals T_{S1} and T_{S2} to be satisfied.

$F_{s1} = \frac{1}{T_{s1}} > 2$ and $F_{s2} = \frac{1}{T_{s2}} > 4$

Fast Fourier Transform（FFT)

- FFT is a fast calculation version of discrete Fourier transform（DFT）.

Discrete Fourier Transform（DFT）

- DSFT with frequency discretization
- In case where $g(n_1, n_2)$が is defined in $N_1 \times N_2$, a finite domain, i.d. 2-dimensional image.

$$g(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} G(k_1, k_2) W_{N_1}^{-k_1 n_1} W_{N_2}^{-k_2 n_2}$$

$$G(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) W_{N_1}^{k_1 n_1} W_{N_2}^{k_2 n_2}$$

Where

$$W_{N_1} = e^{-j2\pi/N_1}, W_{N_2} = e^{-j2\pi/N_2}$$

Thus the values of DFT are sampled ones of DSFT obtained by the intervals uniform intervals of spectrum period/N_1 and N_2.

Periodicity of DFT

$$W_N^{nk} = W_N^{n(k+N)} = W_N^{(n+N)k}$$

に注意すると,

$$G(k_1, k_2) = G(k_1 + N_1, k_2)$$
$$= G(k_1, k_2 + N_2)$$
$$g(n_1, n_2) = g(n_1 + N_1, n_2)$$
$$= g(n_1, n_2 + N_2)$$

- The number of independent points in both of the spatial and frequency domains is N1×N2 and we assume their periodicity and perform calculations.

Exercise Example

- 1 dimensional discrete signal of N points

$$g(n) = \begin{cases} 1, & n = 0 \cdots, L-1 \\ 0, & n = L, L+1, \cdots, N-1 \end{cases}$$

を考える．以下の問いに答えよ．

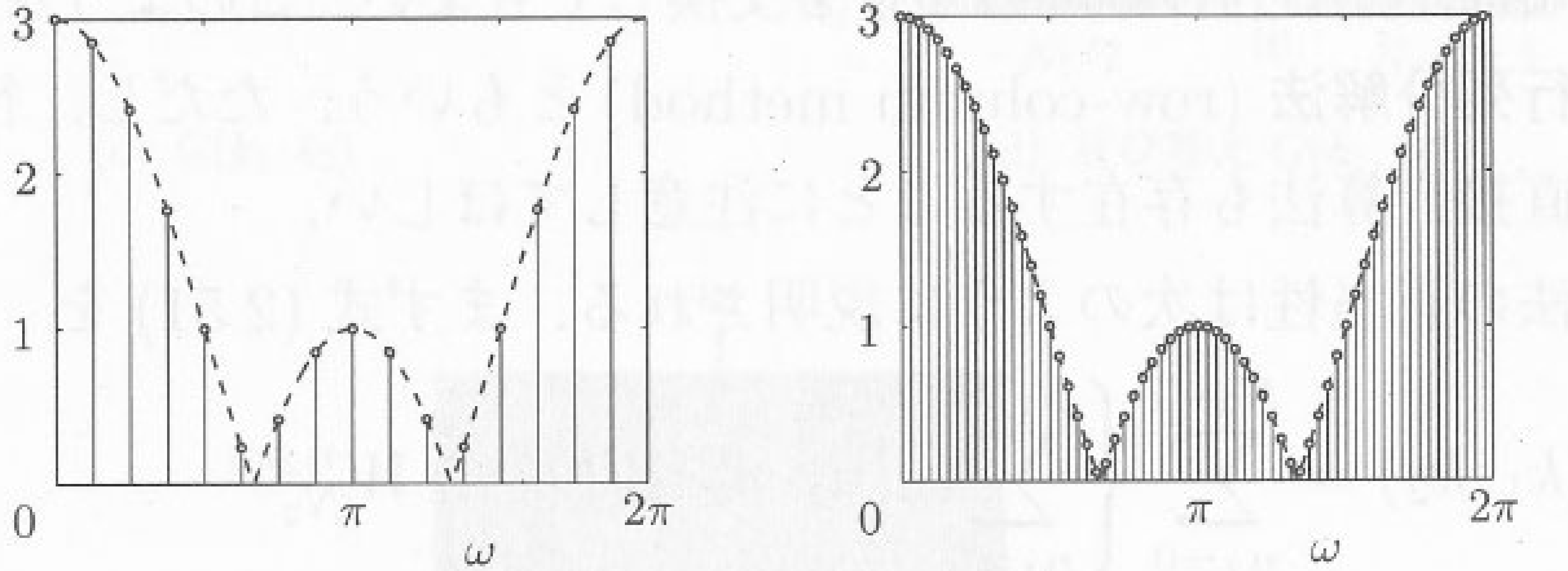
(a) $L = 3$, $N = 16$ として N 点 DFT を求めよ．

(b) $L = 3$, $N = 64$ として N 点 DFT を求めよ．

You can use a calculator to calculate amplitude.

Since (b) takes long time , please do (a).

Answer



(a) $L=3,\ N=16$ (b) $L=3,\ N=64$

- The larger N, the more sufficient sampling density
- For Fourier image analysis, how to make sure to get sufficient sampling density?

Fast Fourier Transform（FFT）

- FFT is to make DFT（a lot of computational cost）faster.
- Without approximation error, it can perform DFT strictly.
- The method which takes advantage of Matrix decomposition method（decomposability of DFT）.

Matrix Decomposition (decomposability of DFT)

- Direct 2D DFT repeats N1×N2 DFT by N1×N2 times.
- In case of matrix decomposition, for horizontal row data perform N1 1D DFT by N2 times, then for column data perform N2 1D DFT by N1 times.

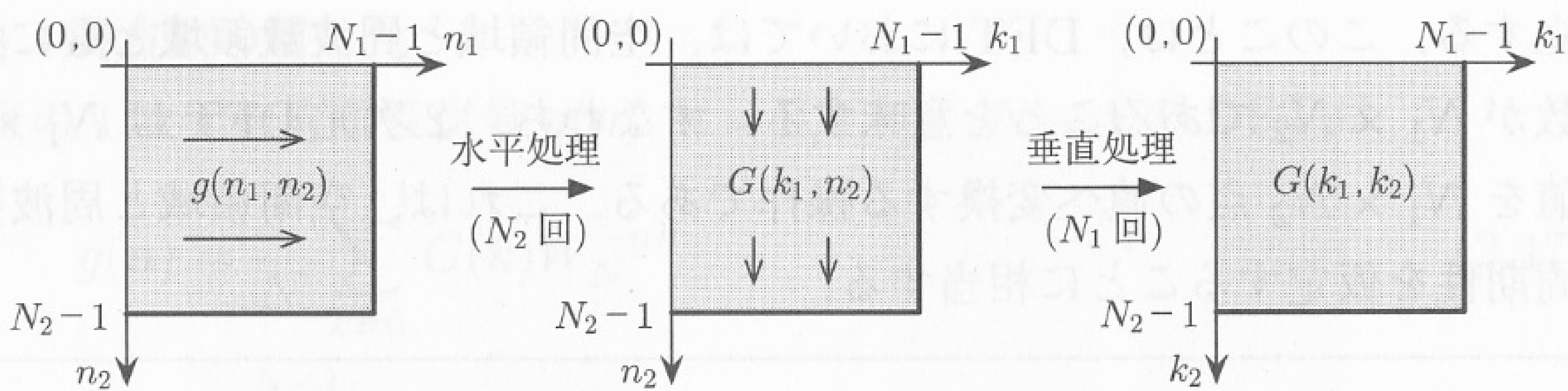


図 **2.16** 行列分解法

Example

- (a): 2D image signal
- (b): for (a), perform DFT horizontally
- (c): for (b), perform DFT vertically.
- (d): By using the periodicity of DFT, put DC component at the center.

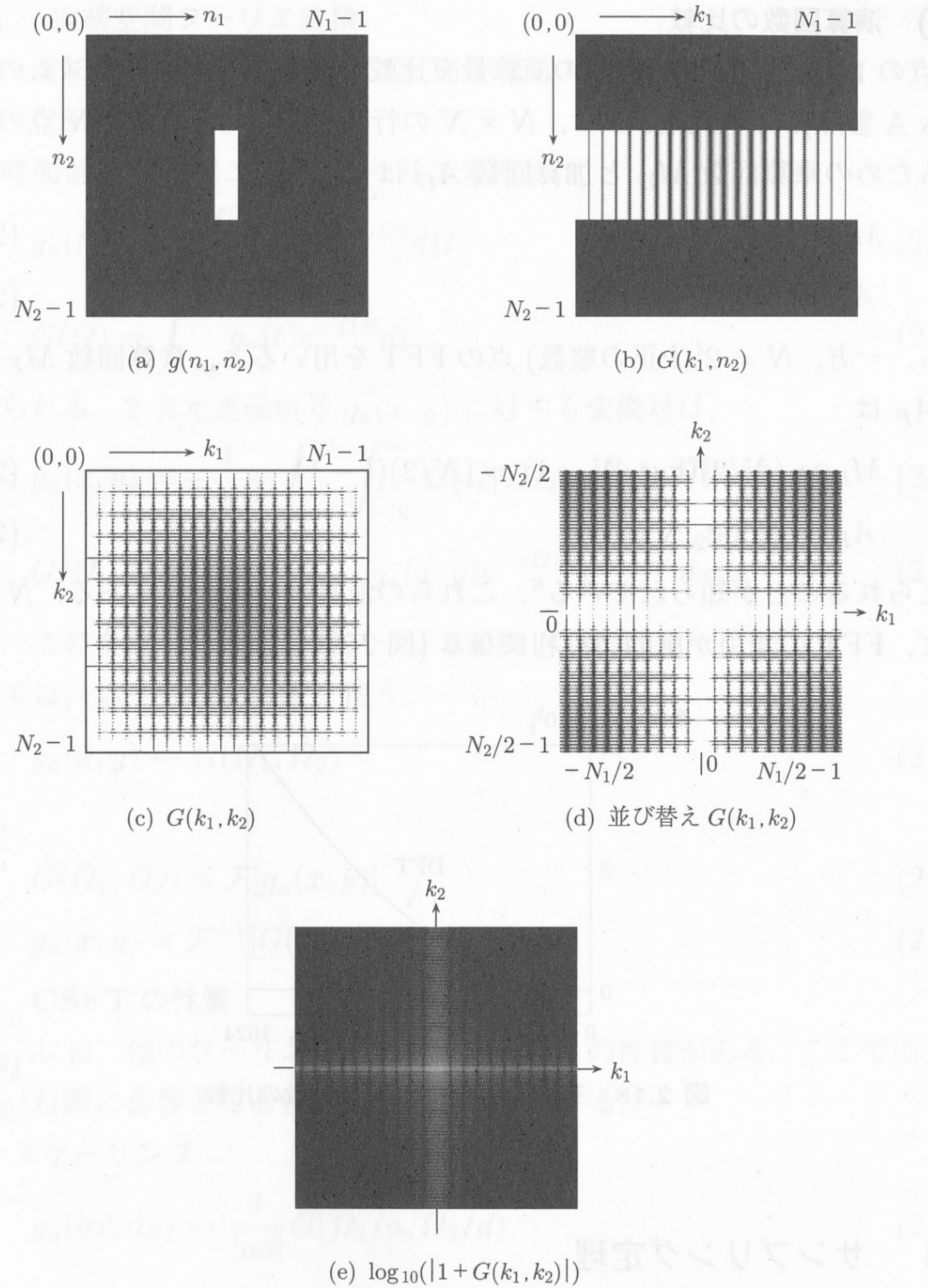
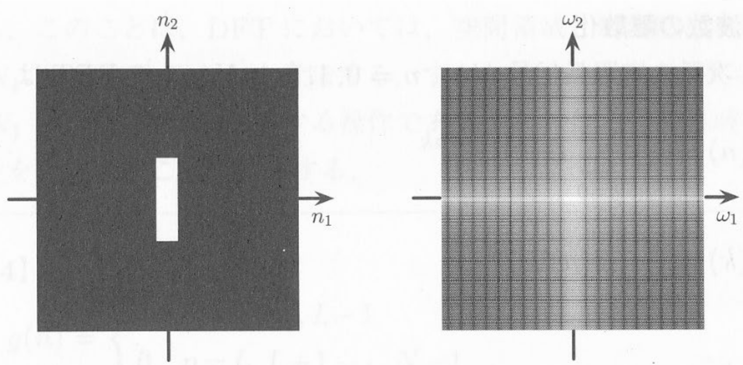


図 2.17　行列分解法による FFT 計算例



Comparison of Operation Number

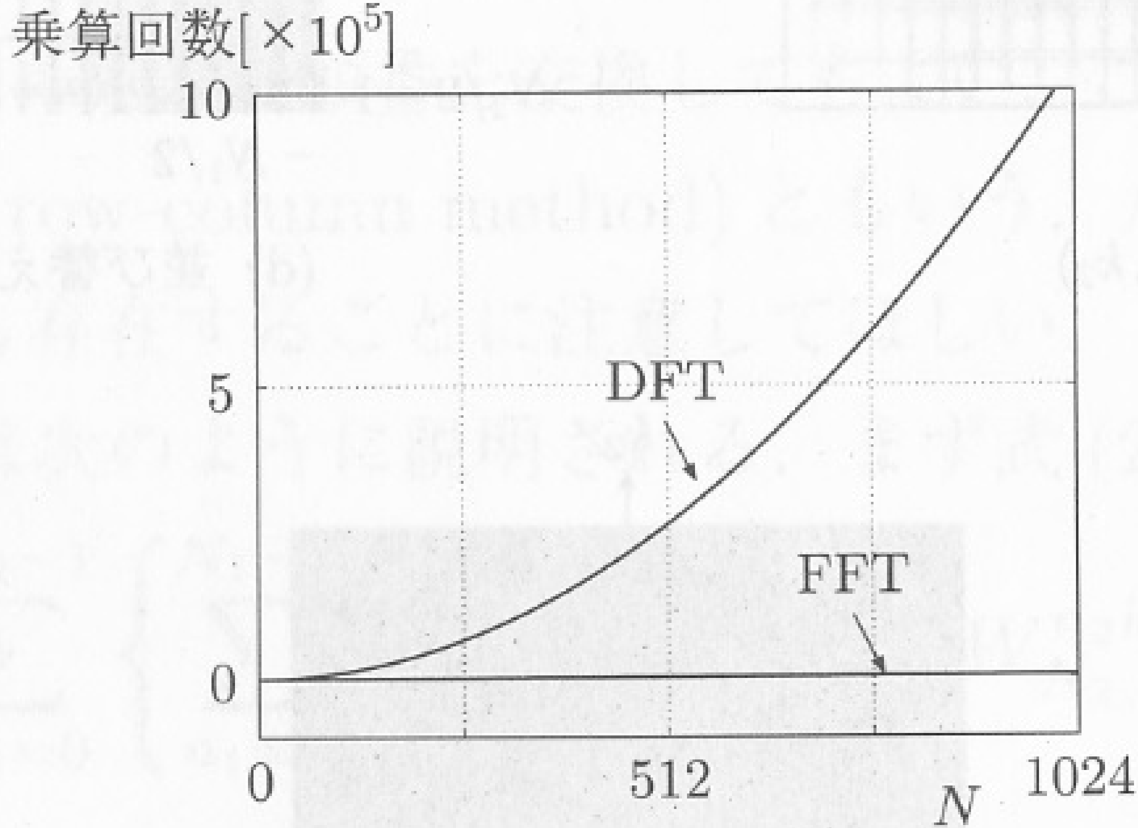


図 **2.18**　DFT と FFT の乗算回数の比較

Sampling Effect

- Sampling generally gives signals distortions (aliasing).

Sampling Theorem

- Theorem that give some condition to avoid the effects of sampling

Sampling Theorem

- Fig.(a): 2D continuous signal’s bandwidth is limited by angular frequency Ωm1 and Ωm2.（No signal exists outside of the limited bandwidth.）
- Fig.(b): Assume rectangular sampling, 2D discrete signal has rectangular periodic spectrum.

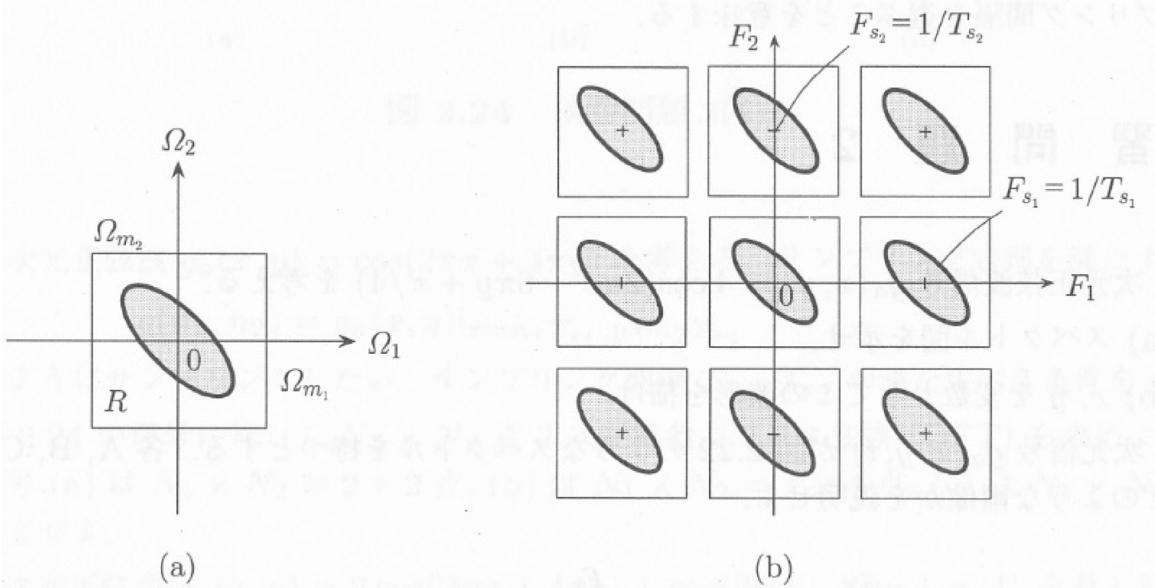


図 2.20　サンプリングの影響

Sampling Theorem

$$F_{s_1} = 1/T_{s_1} > 2F_{m_1}, \text{かつ } F_{s_2} = 1/T_{s_2} > 2F_{m_2}$$

No overlap exists for spectrum.

- Theoretically it is possible to reconstruct perfectly the original signal from sample values by filtering.

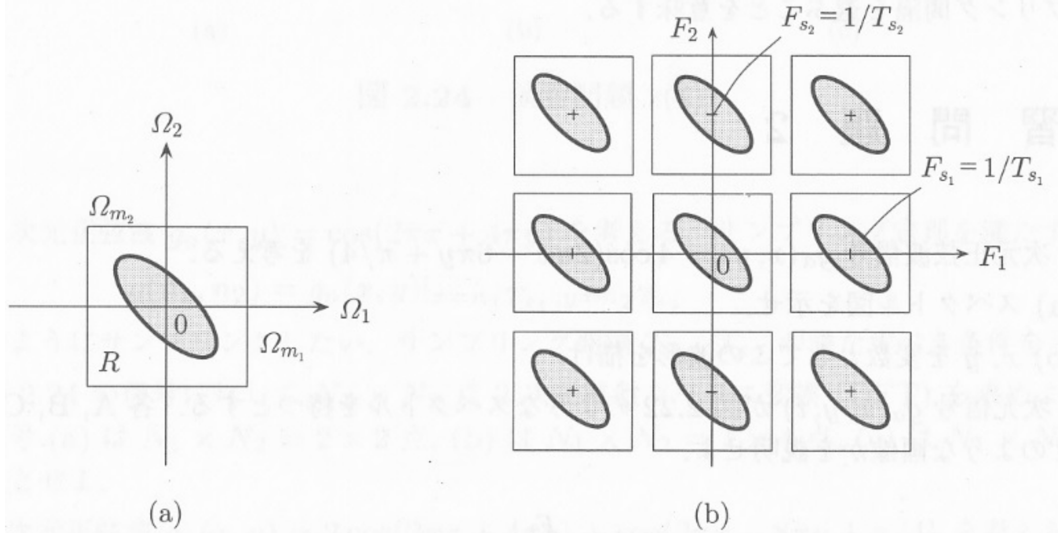
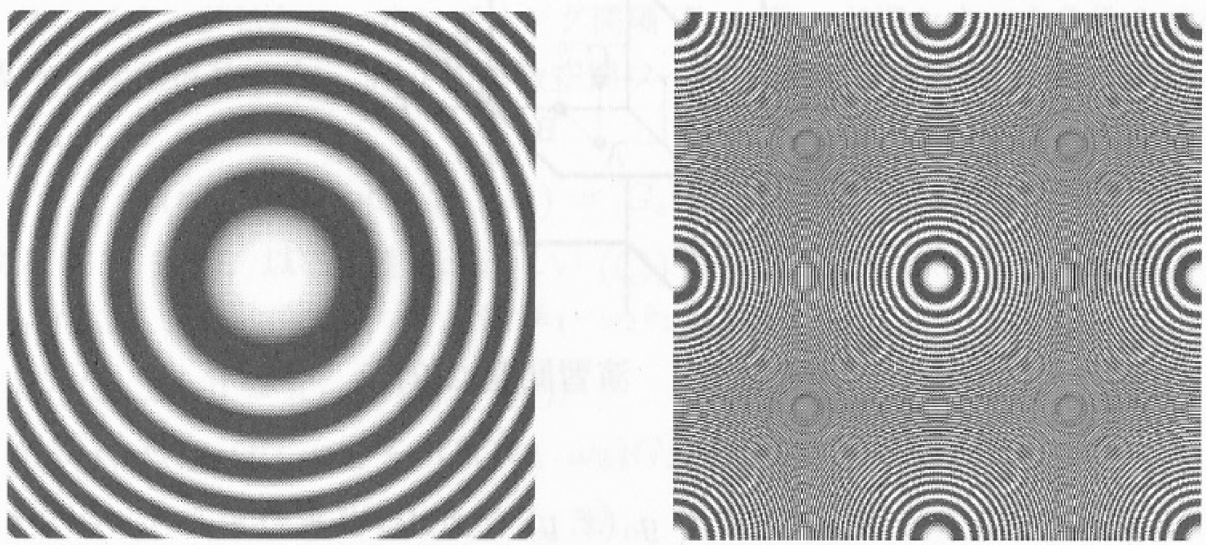


図 2.20　サンプリングの影響

折り返し歪み（Aliasing）

- By sampling without keeping sampling theorem, the spectrums overlap and distort the continuous signal. This distortion (overlap of spectrums) is called aliasing.



(a) エイリアジングなし　　(b) エイリアジングあり

図 **2.21**　エイリアジングの発生例

Example

$$g_a\,(x, y) = \cos(2\pi x + 4\pi y)$$

Calculate the maximum sapling intervals T_{s1} and T_{s2} to keep the sampling theorem.

Answer

- Since the spatial frequency is 1, 2 respectively in the x and y directions, the minimum sampling frequencies are 2, and 4 and their corresponding sampling intervals T_{s1} and T_{s2} are ½ and ¼, respectively.

Basics of Multi-dimensional Filter

- Most of image processing perform filtering to remove or enhance specific frequency components.
- Today we will study about filtering in the spatial domain and frequency domain.

Typical Signals

- 2D sinusoidal wave signal

$$g(n_1, n_2) = A\cos(\omega_1 n_1 + \omega_2 n_2 + \theta)$$

- 2D complex sinusoidal wave signal

$$g(n_1, n_2) = Ae^{j(\omega_1 n_1 + \omega_2 n_2)}$$

- 2D unit sample signal（2D unit impulse)

$$\delta(n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0 \\ 0, & \text{その他} \end{cases}$$

- 2D unit step signal

$$u(n_1, n_2) = \begin{cases} 1, & n_1 \geqq 0 \text{ かつ } n_2 \geqq 0 \\ 0, & \text{その他} \end{cases}$$

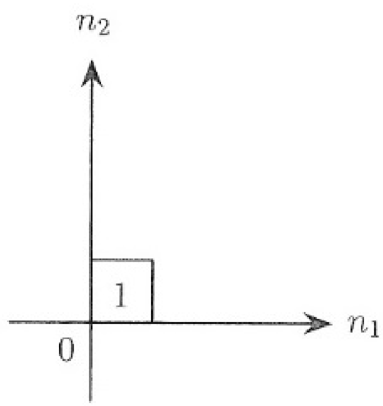
Typical Signals

- 2D unit sample signal（2D impulse signal）

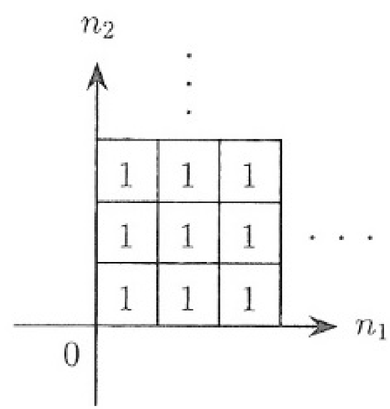
$$\delta(n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0 \\ 0, & \text{その他} \end{cases}$$

- ２D unit step signal

$$u(n_1, n_2) = \begin{cases} 1, & n_1 \geqq 0 \text{ かつ } n_2 \geqq 0 \\ 0, & \text{その他} \end{cases}$$



(a) $\delta(n_1, n_2)$



(b) $u(n_1, n_2)$

Continuous Delta Function δ(t)

$\int_{-\infty}^{\infty} \delta(t)dt = 1$ であり， $\delta(t) = \begin{cases} \infty, t = 0 \\ 0, t \neq 0 \end{cases}$

Example

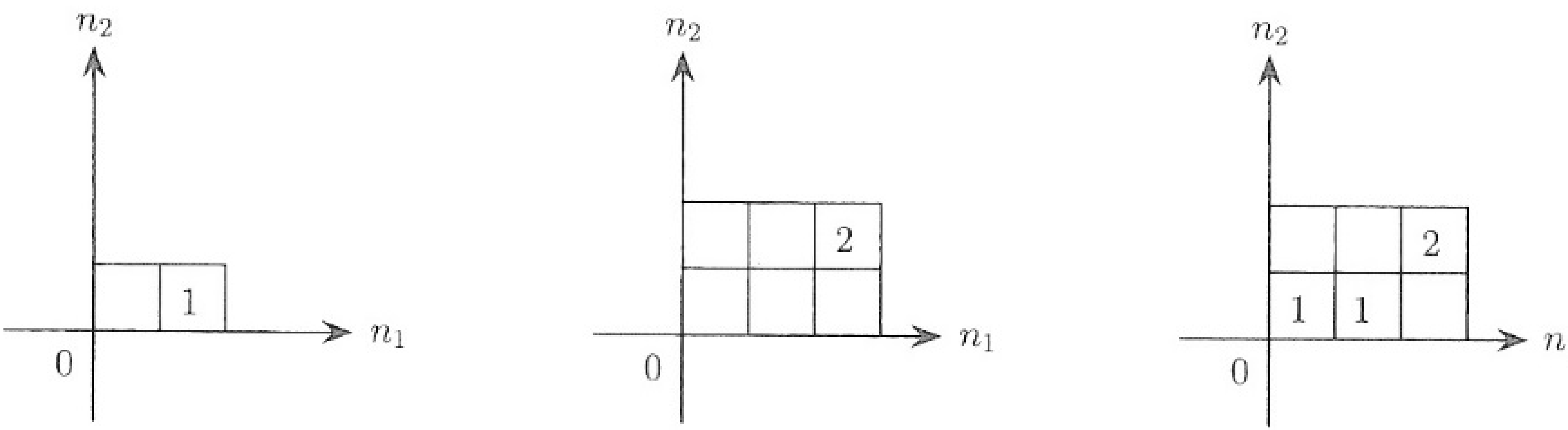
Represent impulse $\delta(n1,n2)$ by 2D unit step signal $u(n1,n2)$.

Answer

$$\delta(n_1, n_2) = u(n_1, n_2) - u(n_1 - 1, n_2) - u(n_1, n_2 - 1) + u(n_1 - 1, n_2 - 1)$$

Example : Impulse Signal

- 2D image is a set of 2D impulse signals.
- Represent the following 2D images by using impulse $\delta(n_1,n_2)$.



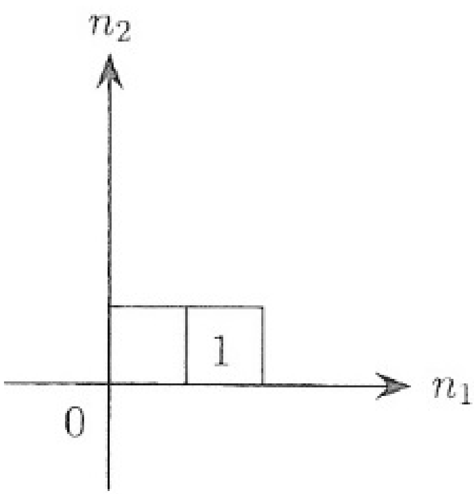
インパルス $\delta(n_1, n_2)$

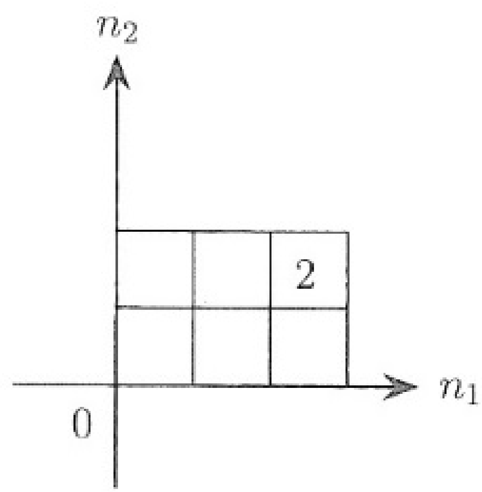
Answers

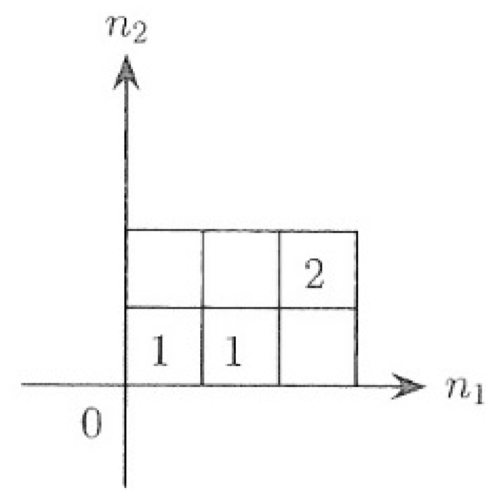
$$g_1(n_1, n_2) = \delta(n_1 - 1, n_2)$$

$$g_2(n_1, n_2) = 2\delta(n_1 - 2, n_2 - 1)$$

$$g(n_1, n_2) = \delta(n_1, n_2) + \delta(n_1 - 1, n_2) + 2\delta(n_1 - 2, n_2 - 1)$$







Representation of 2D Image by Impulse

- By generalization,

$$g(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} g(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)$$

Example Exercise

【例題 **3.2**】 図 3.3 の信号をインパルスを用いて表せ.

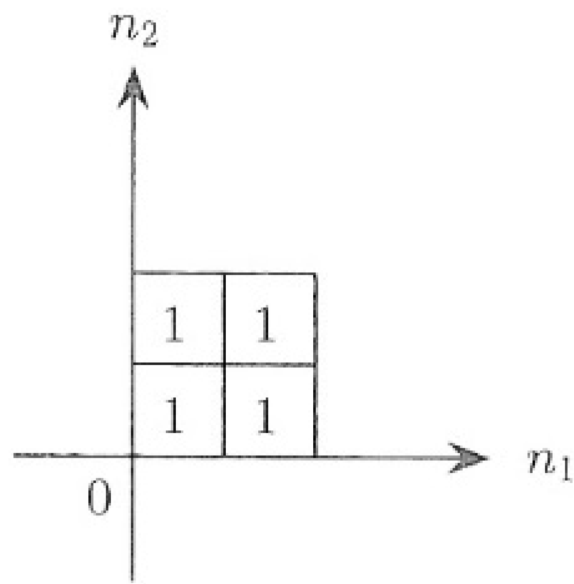


図 **3.3** 例題 3.2

Answer

$$g(n_1, n_2) = \delta(n_1, n_2) + \delta(n_1 - 1, n_2) + \delta(n_1, n_2 - 1) + \delta(n_1 - 1, n_2 - 1)$$

Exercise Example

What is the signal given by

$$g(n_1,n_2) = \sum_{k_1=0}^{\infty} \sum_{k_2}^{\infty} \delta(n_1 - k_1, n_2 - k_2)?$$

Answer

2D unit step signal u(n1,n2)

Separability of Signal

- If a 2D signal is represented by a product of two 1D signals, it is called a separable signal.
- If not, it is called a non-separable signal.

$$g(n_1, n_2) = g_1(n_1) g_2(n_2)$$

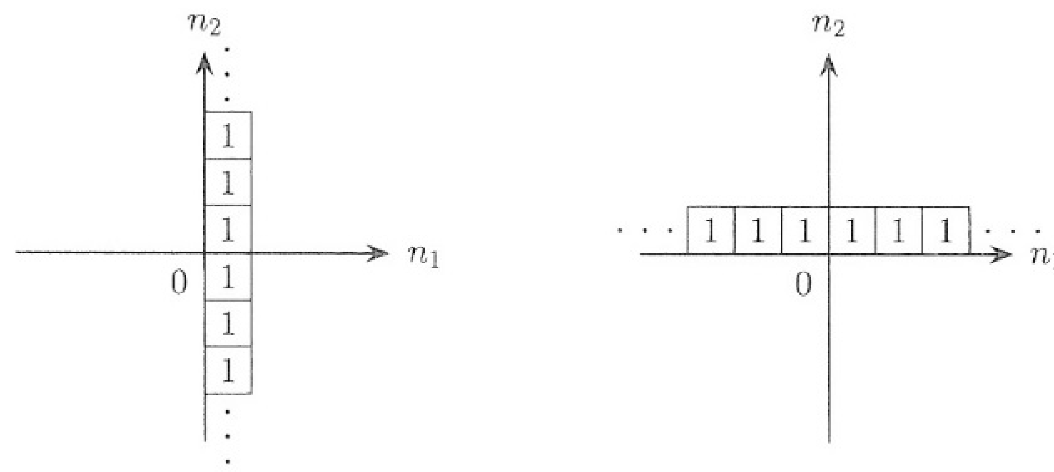
Example : Separability of Signal

- Most general signal is non-separable.
- Are 2D unit impulse signal and 2D unit steps signal separable?

Answer

$$g_1(n_1, n_2) = \delta(n_1)$$

$$g_2(n_1, n_2) = \delta(n_2)$$



(a) $g_1(n_1, n_2) = \delta(n_1)$ (b) $g_2(n_1, n_2) = \delta(n_2)$

See the left figure.

Hence, $\delta(n_1, n_2) = \delta(n_1)\delta(n_2)$

2D unit step signal is also separable.

Example

- Is the signal in Fig. 3.5 separable or non-separable?

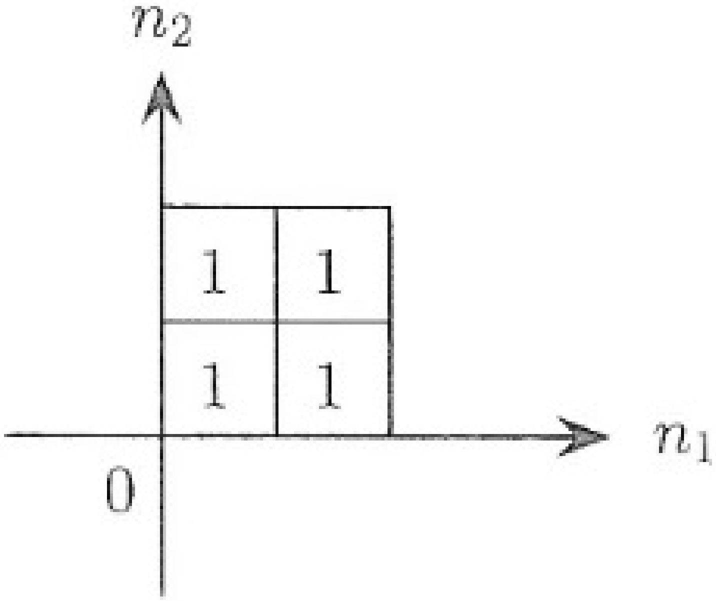


図 **3.5**　例題 3.4

Answer

$$g_1(n_1) = \begin{cases} 1, & n_1 = 0, 1 \\ 0, & \text{その他} \end{cases} \qquad g_2(n_2) = \begin{cases} 1, & n_2 = 0, 1 \\ 0, & \text{その他} \end{cases}$$

とすると，$g(n_1, n_2) = g_1(n_1)g_2(n_2)$ と表現される．