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A New Log-aesthetic Space Curve Based on Similarity Geometry

Kenjiro T. Miura[1], Sho Suzuki[2], R.U. Gobithaasan[3], Shin Usuki[4]

[1]Shizuoka University, miura.kenjiro@shizuoka.ac.jp
[2]Shizuoka University, suzuki.sho.14a@shizuoka.ac.jp
[3]Universiti Malaysia Terengganu, gr@umt.edu.my
[4]Shizuoka University, dsusuki@ipc.shizuoka.ac.jp

Corresponding author: Kenjiro T. Miura, miura.kenjiro@shizuoka.ac.jp

Abstract. A new formulation of log-aesthetic space curves (LASC) is proposed based on the expression of planar log-esthetic curves in the form of similarity geometry. In this new formulation, the slope of the logarithmic torsion graph β is avoided to reduce the complication of LASC generation. It is evident that generalized helix and log-aesthetic magnetic curves are also a part of the new LASC formulation. The final part of this paper details a novel method to generate LASCs with given G^1 Hermite data.

Keywords: Log-aesthetic space curve, Hermite Data, Similarity geometry, Curvature, Torsion
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1 INTRODUCTION

Recently it was found that the log-aesthetic curves(LAC)[8] can be mathematically characterized by using the theory of integrable systems by Inoguchi et al.[5]. They pointed out that LAC can be considered in the framework of similarity geometry, and it is characterized as the similarity geometric analogue of elasticae. Also it can be formulated as a stationary curve with respect to the simplest integrable flow given by the Burgers equation. Hence the theoretical aspects of the researches on LAC have been promoted to the next stage and now it is ready to use these mathematical characterizations for practical applications in CAGD. As one of the applications, in this paper we study about reformulation of the log-aesthetic space curve(LASC). The original definition of LASC has an extra parameter β[7, 12] and its properties are not well understood, especially in terms as a shape parameter as compared to the shape parameter α which determines the type of spiral LAC becomes. In this paper we propose a new formulation of LASC based on similarity geometry without β, which is based on the recent findings of the theory on LAC.

In similarity geometry, we identify figures overlapped by similarity transformations in addition to the Euclidean congruent transformations.This rest of this section reviews the theory of space curves in Euclidian geometry. Let $\boldsymbol{C}(s)$ be a smooth space curve parameterized by an arc length s from the start point of the curve. The unit tangent vector of the curve $\boldsymbol{T}(s)$ is given by a vector $\boldsymbol{C}(s)/ds$ and the unit normal vector $\boldsymbol{N}(s)$ is defined as a unit vector whose direction is in the same direction as the derivative of $\boldsymbol{T}(s)$ with respect to the arc length. The unit binormal vector $\boldsymbol{B}(s)$ is given by the cross product of $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$. The triplet of the unit vector $[\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)]$ set in 3×3 matrix is called a frame. Differentiating this frame we obtain the Frenet-Serret formula as follows[2]:

$$\left[\frac{d\boldsymbol{T}}{ds}(s), \frac{d\boldsymbol{N}}{ds}(s), \frac{d\boldsymbol{B}}{ds}(s)\right] = [\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)] \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \tag{1}$$

In similarity geometry since an arc length is not invariant by similarity transformations, we parametrize a curve as $C(\theta)$ where θ is defined by $d\theta/ds = \kappa(s)$. For the plane curve, θ corresponds to the relative direction angle θ from the start point of the curve. Thus, θ is an invariant parameter in similarity geometry.

Let $\boldsymbol{T}^{sim}(\theta) = d\boldsymbol{T}(s)/d\theta$ be the similarity tangent vector. Then

$$\begin{aligned} \boldsymbol{T}^{sim}(\theta) &= \frac{d\boldsymbol{C}}{d\theta} = \frac{ds}{d\theta}\frac{d\boldsymbol{C}}{ds} \\ &= \frac{1}{\kappa(s)}\boldsymbol{T}(s) = \rho(s)\boldsymbol{T}(s) \end{aligned} \tag{2}$$

Similarly we define $\boldsymbol{N}^{sim}(\theta) = \boldsymbol{N}(s)/\kappa(s)$ and $\boldsymbol{B}^{sim}(\theta) = \boldsymbol{B}(s)/\kappa$ as the similarity tangent and binormal vectors respectively.

Differentiating the similarity frame $[\boldsymbol{T}^{sim}(\theta), \boldsymbol{N}^{sim}(\theta), \boldsymbol{B}^{sim}(\theta)]$, we obtain the similarity Frenet-Serret formula as follows:

$$\left[\frac{d\boldsymbol{T}^{sim}}{d\theta}(\theta), \frac{d\boldsymbol{N}^{sim}}{d\theta}(\theta), \frac{d\boldsymbol{B}^{sim}}{d\theta}(\theta)\right] = [\boldsymbol{T}^{sim}(\theta), \boldsymbol{N}^{sim}(\theta), \boldsymbol{B}^{sim}(\theta)] \begin{bmatrix} -\tilde{\kappa} & -1 & 0 \\ 1 & -\tilde{\kappa} & -\tilde{\tau} \\ 0 & \tilde{\tau} & -\tilde{\kappa} \end{bmatrix} \tag{3}$$

where

$$\tilde{\kappa} = \frac{1}{\kappa^2}\frac{d\kappa}{ds} = -\frac{d\rho}{ds} = -\frac{1}{\rho}\frac{d\rho}{d\theta} \tag{4}$$

$$\tilde{\tau} = \frac{\tau}{\kappa} = \rho\tau \tag{5}$$

$\tilde{\kappa}$ are called similarity curvature and we call $\tilde{\tau}$ as similarity torsion.

2 LOG-AESTHETIC CURVES

The following Theorem was proved by Sato and Shimizu[10] and stated as a theorem by Inogichi[4]:

Theorem 1 *Log-aesthetic curves are characterized as plane curves whose reciprocal similarity curvature $1/\tilde{\kappa}(\theta)$ is a linear function of θ.*

Miura[6] formulated a family of log-aesthetic curves (LAC) which has strict linear Logarithmic Curvature Graph with the slope α and derived a general formula for the radius of curvature as follows (ρ is normalized to be 1 at $s = 0$):

$$\rho(s) = \begin{cases} e^{\lambda s} & \alpha = 0, \\ (\lambda\alpha s + 1)^{\frac{1}{\alpha}} & \alpha \neq 0. \end{cases} \tag{6}$$

$$\theta(s) = \begin{cases} \frac{1-e^{-\lambda s}}{\lambda} & \alpha = 0, \\ \frac{\log(\lambda s+1)}{\lambda} & \alpha = 1, \\ \frac{(\lambda\alpha s+1)^{\frac{\alpha-1}{\alpha}}-1}{\lambda(\alpha-1)} & \text{otherwise.} \end{cases} \tag{7}$$

2.1 Log-aesthetic Space Curves

The radius of torsion μ_{LASC} for LASC is defined by

$$\mu_{LASC} = \begin{cases} ce^{ds} & \beta = 0, \\ (cs + d)^{\frac{1}{\beta}} & \beta \neq 0. \end{cases} \tag{8}$$

in order to preserve the self-affinity property[7] where c,d are arbitrary constants and β is the slope of logarithmic torsion graph which acts as an extra shape parameter. Hence the reciprocal similarity torsion $\tilde{\mu}_{LASC}$ (or the similarity radius of torsion[9]) of LACs is given by

$$\tilde{\mu}_{LASC} = \frac{1}{\tilde{\tau}_{LASC}} = \kappa_{LASC}\mu_{LASC} \tag{9}$$

where

$$\kappa_{LASC} = \begin{cases} e^{-\lambda\theta} & \alpha = 1 \\ (\lambda(\alpha-1)\theta + 1)^{\frac{1}{1-\alpha}} & \alpha \neq 1 \end{cases} \tag{10}$$

Hence, μ_{LASC} is relatively complicated depending on α and β values. When $\beta = 0$,

$$\mu_{LASC} = \begin{cases} c(1-\lambda\theta)^{-\frac{d}{\lambda}} & \alpha = 0, \\ ce^{\frac{d}{\lambda}(e^{\lambda\theta}-1)} & \alpha = 1, \\ ce^{\frac{d}{\lambda\alpha}\{(1+(\alpha-1)\lambda\theta)^{\frac{\alpha}{\alpha-1}}-1\}} & \text{otherwise.} \end{cases} \tag{11}$$

When $\beta \neq 0$,

$$\mu_{LASC} = \begin{cases} (-\frac{c}{\lambda}\log(1-\lambda\theta) + d)^{\frac{1}{\beta}} & \alpha = 0, \\ (\frac{c}{\lambda}(e^{\lambda\theta} - 1) + d)^{\frac{1}{\beta}} & \alpha = 1, \\ \{\frac{c}{\lambda\alpha}((1 + (\alpha-1)\lambda\theta)^{\frac{\alpha}{\alpha-1}} - 1) + d\}^{\frac{1}{\beta}} & \text{otherwise.} \end{cases} \tag{12}$$

It is evident that the similarity radius of LASC's torsion is rather complicated even though the similarity radius of curvature is given by a linear function of θ.

2.2 New Extension of LASC

In this section, we define a family of new space curves by assuming the fact stated in **Theorem 1** and that the reciprocal similarity torsion is given by a linear function of θ. We can determine the radius of torsion as

$\mu = 1/\tau$ instead of explicitly expressing the torsion τ as stated in Eq.(8). Assuming $1/\tilde{\tau} = \tilde{\mu} = a\theta + b$ in Eq.(5), μ is given by

$$\mu = \rho(a\theta + b) \tag{13}$$

where a and b are arbitrary constants. From Eqs.(6) and (7), when $\alpha = 0$,

$$\mu = e^{\lambda s}(a\frac{1 - e^{-\lambda s}}{\lambda} + b) = (\frac{a}{\lambda} + b)e^{\lambda s} - \frac{a}{\lambda} \tag{14}$$

When $\alpha = 1$,

$$\mu = (\lambda s + 1)(\frac{a}{\lambda}\log(\lambda s + 1) + b) \tag{15}$$

Otherwise

$$\begin{aligned} \mu &= (\lambda\alpha s + 1)^{\frac{1}{\alpha}}(a\frac{(\lambda\alpha s + 1)^{\frac{\alpha-1}{\alpha}} - 1}{\lambda(\alpha - 1)} + b) \\ &= \frac{a}{\lambda(\alpha - 1)}(\lambda\alpha s + 1) + (b - \frac{a}{\lambda(\alpha - 1)})(\lambda\alpha s + 1)^{\frac{1}{\alpha}} \\ &= a_0(\lambda\alpha s + 1) + b_0(\lambda\alpha s + 1)^{\frac{1}{\alpha}} \end{aligned} \tag{16}$$

where $a_0 = a/\lambda(\alpha - 1)$ and $b_0 = b - a/\lambda(\alpha - 1)$.

The original definition stated in Eq.(8) has an extra parameter β and its properties are not well understood, especially in terms as a shape parameter as compared to the shape parameter α which determines the type of spiral LAC becomes. This new definition is similar to the original representation if $\beta = \alpha$, but still there are some slight differences. Figure 1 shows a comparison of LASCs defined by our new formulation and defined traditionally. The values of shape parameters are $\alpha = \beta = -0.5$ and their curvatures are given by

$$\kappa(s) = 0.04s^2 \tag{17}$$

The torsion τ_n of the newly defined LASC is

$$\tau_n = \frac{0.06s^2}{0.16\theta + 0.81} \approx \frac{0.06s^2}{0.213s^3 + 0.81} \tag{18}$$

The torsion τ_t of the traditionally defined LASC is

$$\tau_t = 0.06s^2 \tag{19}$$

It is evident from this numerical example that both log-aesthetic space curves behave differently. Even though their curvature distributions are the same but their torsion distributions start from 0 at the origin and gradually increases differently as shown in Fig.2.

3 INPUT OF LOG-AESTHETIC SPACE CURVE

In this section, we propose to simultaneously specify the endpoints and its tangent vectors for the proposed LASC thus satisfying G^1 Hermite data. To note, Yoshida and Saito[12] used a cone to determine the boundary conditions to reduce two degrees of freedom, i.e. scaling and rotation about the cone axis as shown in Fig.3. However, we employ a formula to calculate the tangent direction at the endpoints and optimize the distance of the endpoints by taking advantage of scaling to reduce one degree of freedom.

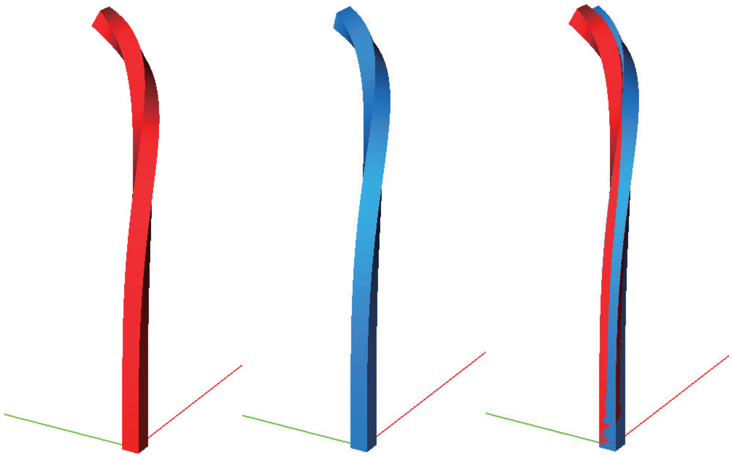


Figure 1: From left to right, LASC by our new definition, traditional LASC, both rendered at the same coordinate.

Without loss of generality, we assume that the slope α of the logarithmic curvature graph is $\alpha = 0$ or $\alpha = 1$. Since the input of the LASC is not straightforward, we simplify our problem by letting the torsion τ of LASC given by $\tau = \kappa/b$. Hence, we assume that $a = 0$ in Eq.(13). For a fixed Frenet frame at the start point, b is automatically determined, but we still have two parameters λ and the rotation angle about the tangent vector at the start point. In the implementation of the proposed algorithm, we use $c = 1/b$ instead of b because a plane curve can be treated as a regular case when $c = 0$.

When τ/κ is a constant, the space curve is called a generalized helix or a cylindrical helix[3]. The tangnet lines of the curve make a constant angle with a fixed direction as shown in Fig.4, where the tangnet lines of the curve make a constant angle θ with the z axis. Hence the new formulation of LASC proposed in this paper includes a special case of the generalized helix; primarily it includes log-aesthetic magnetic curves[11]. Suppose $\tau(s) = c\,\kappa(s)$, then the Frenet-Serret formula is given by

$$\left[\frac{d\boldsymbol{T}}{ds}(s), \frac{d\boldsymbol{N}}{ds}(s), \frac{d\boldsymbol{B}}{ds}(s)\right] = \kappa(s)[\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)] \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -c \\ 0 & c & 0 \end{bmatrix} \tag{20}$$

It is well known that the system of differential equation above can be solved [1] by letting $\boldsymbol{T}(0) = (1, 0, 0)$, $\boldsymbol{N}(0) = (0, 1, 0)$ and $\boldsymbol{B}(0) = (0, 0, 1)$. Its solution is given by

$$\begin{aligned} \boldsymbol{T}(s) &= \frac{1}{B}(c^2 + \cos(\sqrt{B}A(s)), \sqrt{B}\sin(\sqrt{B}A(s)), c - c\cos(\sqrt{B}A(s))) \\ \boldsymbol{N}(s) &= \frac{1}{B}(-\sqrt{B}\sin(\sqrt{B}A(s)), B\cos(\sqrt{B}A(s)), c\sqrt{B}\sin(\sqrt{B}A(s))) \\ \boldsymbol{B}(s) &= \frac{1}{B}(c - c\cos(\sqrt{B}A(s)), -c\sqrt{B}\sin(\sqrt{B}A(s)), 1 + c^2\cos(\sqrt{B}A(s))) \end{aligned} \tag{21}$$

where $A(s) = \int_0^s \kappa(s)ds$ and $B = 1 + c^2$. If the curve is in the standard form, then from Eq.(7) $A(s)$ can be

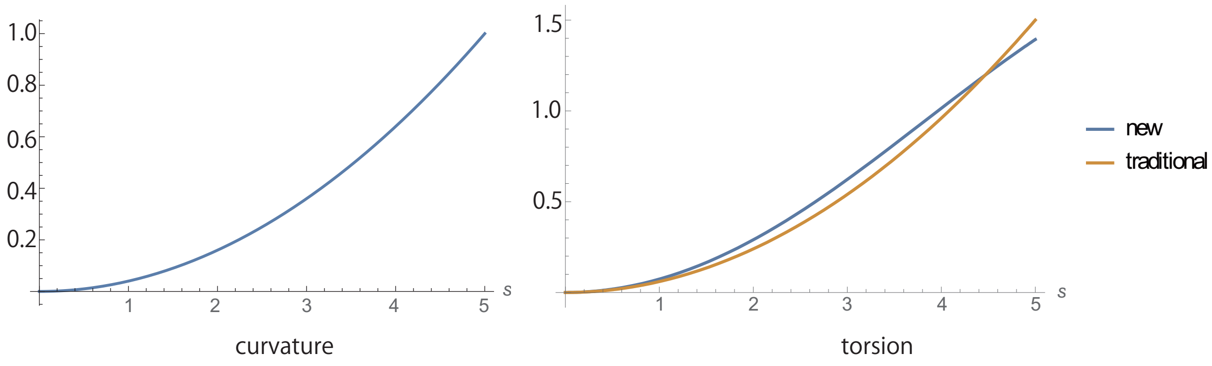


Figure 2: Curvature and torsion distributions.

derived as follows

$$A(s) = \frac{(\lambda \alpha s + 1)^{\frac{\alpha-1}{\alpha}} - 1}{\lambda(\alpha - 1)} \tag{22}$$

3.1 Initial Value Estimation

We have to estimate λ in Eq.(6) and $c = 1/b$ in Eq.(13) to control the position of endpoints. At first, we estimate c as follows. Let the tangent vectors $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$ at the endpoints denoted as $(1, 0, 0)$ and (u, v, w), respectively. From Eq.(21), $\boldsymbol{t}_1$ is given by

$$u = \frac{1}{(1+c^2)}(c^2 + \cos(\sqrt{B}A(s))) \tag{23}$$

$$w = \frac{1}{(1+c^2)}(c - c\cos(\sqrt{B}A(s))) \tag{24}$$

Hence

$$u + \frac{w}{c} = \frac{1+c^2}{1+c^2} = 1 \tag{25}$$

Then

$$c = \frac{w}{1-u} \tag{26}$$

If we fix the Frenet frame at the start point, the coefficient c of the torsion τ and curvature κ can be determined. For a wider boundary condition, we should preserve one degree of freedom by allowing the rotation about the tangent vector at the start point. The initial value of c can be estimated by stating the normal vector at the start point of the curve, for example $(0, 1, 0)$.

We still have to estimate the initial λ value if the binormal vector at the start point is given by $(0, 0, 1)$. We project the boundary condtion in the xy plane and generate a log-aesthetic curve in the standard form to estimate the initial λ.

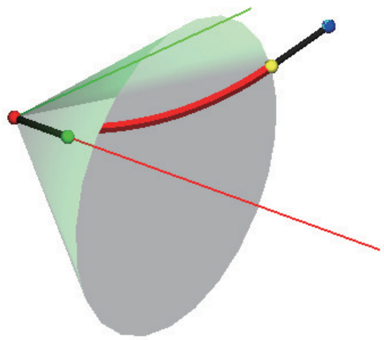


Figure 3: A cone whose central axis is in the tangent direction at the start point in which the apex is the start point.

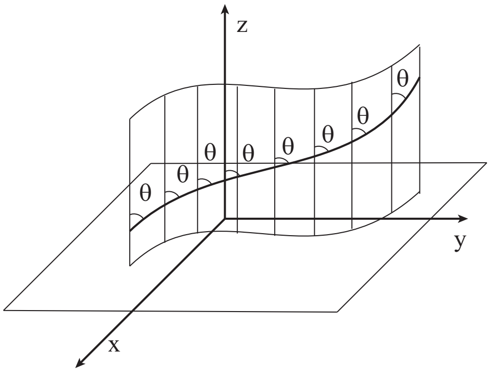


Figure 4: The tangent lines of a generalized helix make a constant angle with a fixed direction.

In the xy plane, from Eq.(21) the tangent vector at the start point is given by $(1, 0)$ and the unit tangent vector at the end point is obtained by projecting the tangent vector on the xy plane and normalizing as follows:

$$\boldsymbol{T}_{2D}(s) = \frac{1}{B}(c^2 + \cos(\sqrt{B}A(s)), \sqrt{B}\sin(\sqrt{B}A(s))) \tag{27}$$

Note that in the above expression of arc length s for the space curve is no more an arc length for the 2D curve. The first derivative of $\boldsymbol{T}_{2D}(s)$ is given by

$$\frac{d\boldsymbol{T}_{2D}(s)}{ds} = \frac{\kappa}{B}(-\sqrt{B}\sin(\sqrt{B}A(s)), B\cos(\sqrt{B}A(s))) \tag{28}$$

Therefore the curvature $\kappa_{2D}(s)$ of the 2D curve is given by

$$\kappa_{2D}(s) = \frac{|\boldsymbol{T_{2D}} \times \frac{d\boldsymbol{T_{2D}(s)}}{ds}|}{|\boldsymbol{T}_{2D}|^3} \tag{29}$$

$$= \frac{\kappa(1+c^2)^2}{(c^4 + c^2(-\cos^2(\sqrt{B}A(s)) + 2\cos(\sqrt{B}A(s)) + 1) + 1)^{\frac{3}{2}}}(1 + c^2\cos(\sqrt{B}A(s))) \tag{30}$$

Note that $\kappa_{2D}(0) = \kappa(0)$ since $\cos(\sqrt{B}A(s)) = 1$. It means that the curvature of the 2D curve and 3D curve are identical at the start point. Using equation above, we can express $\kappa(s)$ at the end point by $\kappa_{2D}(s)$ and estimate λ. The total arc length of the 3D curve in the standard form is estimated by the initial λ and Eq.(23).

3.2 Better Initial Value Estimation

Let us rotate the tangent vector given in Eq.(21) about $(0,1,0)$ by angle α where $\tan\alpha = c$. The new tangent vector $\boldsymbol{T}_\alpha$ is given by

$$\boldsymbol{T}_\alpha(s) = \frac{1}{B}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}\begin{bmatrix} c^2 + \cos(\sqrt{B}A(s)) \\ \sqrt{B}\sin(\sqrt{B}A(s)) \\ c - c\cos(\sqrt{B}A(s)) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos(\sqrt{B}A(s)) \\ \cos\alpha\sin(\sqrt{B}A(s)) \\ \sin\alpha \end{bmatrix} \tag{31}$$

The expression above indicates that the scalar product of vector $(0,0,1)$ with the tangent vector of the curve is always a constant $\sin\alpha$ and the angle between the z axis and the tangent vector is constant. This is a well known property of the generalized helix [3]. Suppose the curvature of the 2D curve projected on the xy plane is κ_{2D}, then the curvature κ of the 3D curve is given by

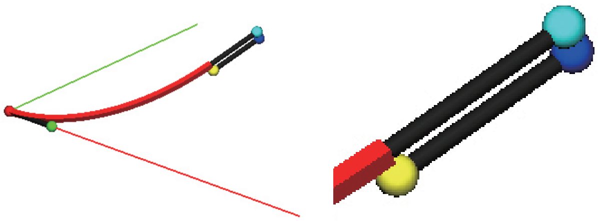
$$\kappa = \frac{\kappa_{2D}}{1+c^2} \tag{32}$$

It is the curvature of an ordinary helix[3]. Therefore, at first we estimate c, then using c we project the boundary condition on the xy plane. First, we generate planar log-aesthetic curve, then we can estimate an initial λ for the LASC. Note that the curvature at the start point of LASC in the standard form is assumed to be equal to 1, thus we can use the same λ value of planar log-aesthetic curve as the initial value of λ for LASC since the $1 + c^2$ is constant for any s.

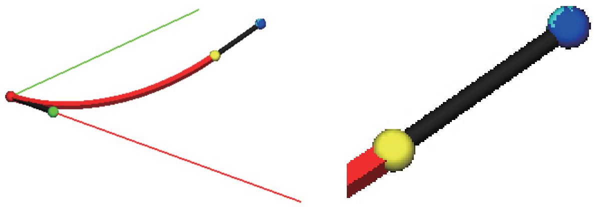
We then employ Downhill Simplex method to minimize the difference between the end point which was given as boundary condition and the end point of the generated LASC by changing λ and the rotation angle θ of the tangent vector at the start point. Since we are searching in two dimensional space, thus we need three set initial values of λ and θ. The vector $\boldsymbol{d}$ in the projected direction from a 3D curve to a 2D curve should make the same angle to both of the tangent vectors $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$, i.e.

$$\boldsymbol{d} \cdot \boldsymbol{t}_0 = \boldsymbol{p} \cdot \boldsymbol{t}_1 \tag{33}$$

Hence $\boldsymbol{d}$ should be on the plane $\boldsymbol{P}$ passing the origin and including $\boldsymbol{t}_0 \times \boldsymbol{t}_1$ and $(\boldsymbol{t}_0 + \boldsymbol{t}_1)/|\boldsymbol{t}_0 + \boldsymbol{t}_1|$. The projected curve will be planar ($c = 0$) when $\boldsymbol{d}$ is in the same direction as $\boldsymbol{t}_0 \times \boldsymbol{t}_1$ since both of the tangent vectors are projected on the plane perpendicular to plane $\boldsymbol{P}$. When $\boldsymbol{d}$ is in the same direction as $(\boldsymbol{t}_0 + \boldsymbol{t}_1)/2$, $c = \tan\alpha$ will be the largest value where α is an angle between the tangent vector and the projection plane. If the position of the end point cannot be reached by the largest value of c, we cannot generate a LASC satisfying given the boundary conditions. However, we can utilize $\boldsymbol{d}_0 = \boldsymbol{t}_0 \times \boldsymbol{t}_1$ and $\boldsymbol{d}_1 = (\boldsymbol{t}_0 + \boldsymbol{t}_1)/|\boldsymbol{t}_0 + \boldsymbol{t}_1|$ and obtain minimum and maximum rotation angles of θ. We may choose three angles between the minimum



Initial curve



Optimized curve

Figure 5: A LASC generated for a given boundary condition

and maximum, for example θ_{min}, $(3\theta_{min} + \theta_{max})/4$ and $\theta_{max}/2$ and estimate λ for each rotation angle by projection.

Figure 5 shows the generated LASC. In this example, the initial estimation using θ_{min} is relatively accurate with given endpoints location as $(0, 0, 0)$ and $(3, 2, 1)$. The initial distance between the given and calculated end point is 0.171 After optimization the distance between given and calculated end point becomes less than $1.e - 6$. The figures on the right shows the close up at the end point. After optimization the curve reaches the center of the end point accurately. The processing time is about 200 msec on a Core i7-6700 3.4GHz processor to reach to the precision of $1.e - 6$. Note that the tangent vector at the end point matches exactly as the given boundary condition at the initial estimation.

4 CONCLUSIONS

A new LASC formula has been derived based on the formulation of log-aesthetic plane curve expressed in terms of similarity geometry. We have utilized that the slope α of the logarithmic curvature graph to determine the slope of logarithmic torsion graph β. This new formulation of LASC include generalized helix and log-aesthetic magnetic curves. A improved method to generate LASC satisfying given G^1 Hermite data is also presented.

Future works include the extension of LASC for $a \neq 0$ and obtaining its drawable region. It is also worth investigating on the properties of similarity torsion, which is invariant under similarity geometry and how it influences aesthetic design.

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ORCID

Kenjiro T. Miura, http://orcid.org/0000-0001-9326-313
Sho Suzuki, http://orcid.org/0000-0002-2607-7204
R.U. Gobithaasan, http://orcid.org/0000-0003-3077-8772
Shin Usuki, http://orcid.org/0000-0003-1165-5507

REFERENCES

[1] Chiba, H.: Introduction to vector analysis and geometry. Gendai-Sugakusha, 2007.

[2] Chou, K.S.; Qu, C.: Motions of curves in similarity geometries and Burgers-mKdV hierarchies. Chaos, Solitons & Fractals, 19(1), 47–53, 2004.

[3] do Carmo, M.P.: Differential Geometry of Curves and Surfaces. Prentice-Hall, 1976.

[4] Inoguchi, J.: Attractive plane curves in differential geometry. In Y. Dobashi; H. Ochiai, eds., Mathematical Progress in Expressive Image Synthesys III, 121–135. Springer, 2016.

[5] Inoguchi, J.; Kajiwara, K.; Miura, K.T.; Sato, M.; Schief, W.K.; Shimizu, Y.: Log-aesthetic curves as similarity geometric analogue of Euler's elasticae. Computer Aided Geometric Design, 61(3), 1–5, 2018.

[6] Miura, K.T.: A general equation of aesthetic curves and its self-affinity. Computer-Aided Design and Applications, 3(1-4), 457–464, 2006.

[7] Miura, K.T.; Fujisawa, M.; Sone, J.; Kobayashi, K.G.: The aesthetic space curve. In Humans and Computers 2006, 101–106. University of Aizu, 2006.

[8] Miura, K.T.; Gobithaasan, R.U.: Aesthetic design with log-aesthetic curves and surfaces. In Y. Dobashi; H. Ochiai, eds., Mathematical Progress in Expressive Image Synthesys III, 107–119. Springer, 2016.

[9] Miura, K.T.; Gobithaasan, R.U.; Suzuki, S.; Usuki, S.: Reformulation of generalized log-aesthetic curves with Bernoulli equations. Computer-Aided Design and Applications, 13(2), 265–269, 2016.

[10] Sato, M.; Shimizu, Y.: Log-aesthetic curves and Riccati equations from the viewpoint of similarity geometry. JSIAM Letters, 7(1), 21–24, 2015.

[11] Wo, M.S.; Gobithaasan, R.U.; Miura, K.T.: Log-aesthetic magnetic curves and their application for CAD systems. Mathematical Problems in Engineering, 2014(504610), 1–16, 2014.

[12] Yoshida, N.; Fukuda, R.; Saito, T.: Log-aesthetic space curve segments. In SIAM/ACM Joint Conference on Geometric and Physical Modeling 2009, 35–46, 2009.