Input of Compound-rhythm Log-aesthetic Curves

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Abstract

This paper proposes a method of inputting a planar log-aesthetic curve with compound-rhythm using four control points. The log-aesthetic curve does not exhibit any undulations of curvature as its curvature increases or decreases monotonically and it is suitable for practical product design. We report an method to input the compound-rhythm log-aesthetic curve made up of two log-aesthetic curve segments connected with C^3 continuity.

Keywords: log-aesthetic curve, compound-rhythm curve, logarithmic curvature graph

1 Introduction

Harada et al. proposed "aesthetic curves" as those whose logarithmic distribution diagram of curvature (LDDC) can be approximated by a straight line [Har97, Har97]. Miura derived analytical solutions of the curves whose logarithmic curvature graph (LCG), an analytical version of the LDDC, is strictly given by a straight line and proposed these solutions as general equations of aesthetic curves [Miu06]. Furthermore, Yoshida and Saito analyzed the properties of the curves and developed a new method for the interactive generation of a curve segment by specifying two end points and their tangent vectors with three so-called control points as well as the slope α of the straight line of the LCG [YS06]. The value of α is preserved as a parameter that can be changed by the designer of the curve since it was first suggested to be closely related to the impression of the curve [Har97]. In this paper, we call the curves expressed by the general equations of aesthetic curves the log-aesthetic curve. We use "log-aesthetic" instead of "aesthetic" to clarify that we are dealing with a specific type of curve[1].

The log-aesthetic curve includes the logarithmic (equiangular) spiral ($\alpha = 1$), the clothoid curve ($\alpha = -1$), the circle involute ($\alpha = 2$), and Nielsen's spiral ($\alpha = 0$). Its curvature increases or decreases monotonically and its evolute is also given by another log-aesthetic curve. This means that it does not exhibit any undulations of curvature. It is possible to generate and deform log-aesthetic curves even if they are expressed by integral forms using their unit tangent vectors as integrands ($\alpha \neq 1, 2$), and they are expected to be used in practical product design, e.g., in the design of cars. However, their input method proposed using three control points [YS06] can generate only a log-aesthetic curve segment and cannot generate a curve with compound-rhythm. The concept of the rhythm of the curve was introduced by [Har97] and the compound-rhythm means that the curve consists of two segments, which are log-aesthetic curves with different values of α.

[HMY98] defined the log-aesthetic curve with compound-rhythm, or the compound-rhythm log-aesthetic curve that consists of two log-aesthetic curve segments with different α values; these segments are connected with continuity of curvature, and the derivative of curvature is continuous between them. The vertical axis of the LCG measures $\log(ds/d(\log\rho))$ and its continuity depends on that of $ds/d\rho$. Since we assume that the LCG of the compound-rhythm curve is continuous, the curve should have C^3 continuity. They

[1]The term "log-aesthetic" was first coined by Prof. Calro Séquin at the University of California, Berkeley in discussions at the International CAD Conference and Exhibition 2007 held in Hawaii.

noted that the compound-rhythm log-aesthetic curve and its approximations have been used frequently for car body design at car design factories in Europe, especially "carrozzeria" in Italy, and it can be regarded as a very important curve for car styling design. Hence, in this paper we propose a method to input compound-rhythm log-aesthetic curves.

There are two typical ways to input a curve: one is to specify passing points, and the other is to specify control points. Due to the ease of input and good controllability, we studied about the input of the curve with control points. The log-aesthetic curve is generally defined in an integral form and it is not straightforward to locate its end at a specific position.

The rest of this paper consists of the following sections. Section 2 reviews the criteria of the quality of a curve and previous research on the log-aesthetic curve. Section 3 defines the formulae of the compound-rhythm log-aesthetic curve. Section 4 presents a method to input a curve using four control points and shows curve examples generated by the method. The final section concludes the paper and discusses future work.

2 Compound-rhythm Log-aesthetic Curve

The formulae of the log-aesthetic curves are generally classified into two cases [MSYK05]: one where $\alpha \neq 0$ and another where $\alpha = 0$. The curve is specified as Nielsen's spiral if $\alpha = 0$ and it is somehow exceptional. If $\alpha = 1$, the curve is the logarithmic spiral and it is not necessary to use an integral form to define the curve. Therefore, in this section we assume $\alpha \neq 0, 1$. Even if α is equal to 0 or 1, we can deal with them similarly.

2.1 Formula of Log-aesthetic Curve

In case of $\alpha \neq 0$, the following equation is satisfied between the radius of curvature ρ and the arc length s of the curve:

$$\rho^{\alpha} = cs + d \tag{1}$$

where c and d are constants. The relationship between the arc length s and the direction angle θ is given by:

$$\theta = \frac{1}{c}\frac{\alpha}{\alpha-1}(cs+d)^{\frac{\alpha-1}{\alpha}} + \theta_e \tag{2}$$

where θ_e is a constant (angle) determined by the direction angle at $s = 0$. We assume that if the curve is turning left, its curvature is positive. Then in the complex plane the point $P(s)$ on the curve is given by:

$$P(s) = P_0 + \int_0^s \exp\left\{ i\left(\frac{1}{c}\frac{\alpha}{\alpha-1}(cs+d)^{\frac{\alpha-1}{\alpha}} + \theta_e\right)\right\} ds \tag{3}$$

where $i = \sqrt{-1}$, and P_0 is the start point of the curve.

2.2 Formula of Compound-rhythm Curve

The compound-rhythm curve consists of two log-aesthetic curve segments and for two given values of α the relationship between the radius of curvature ρ and the arc length s is expressed by:

$$\rho = \begin{cases} (c_0 s + d_0)^{\frac{1}{\alpha_0}} & (0 \leq s \leq s_c) \\ (c_1 s + d_1)^{\frac{1}{\alpha_1}} & (s_c < s \leq s_l) \end{cases} \tag{4}$$

where s_c is the curve length from the start point to the connection point of the segments and α, c and d are different from the connection point, and s_l is the total length of the curve.

Based on the above formulations, at the parameter range $0 \leq s \leq s_c$ the compound-rhythm log-aesthetic curve $P(s)$ is given by:

$$P_0 + \int_0^s \exp\left\{ i\left(\frac{1}{c_0}\frac{\alpha_0}{\alpha_0-1}(c_0 s+d_0)^{\frac{\alpha_0-1}{\alpha_0}} + \theta_{e0}\right)\right\} ds \tag{5}$$

In the parameter range $s_c < s \leq s_l$,

$$P_c + \int_{s_c}^s \exp\left\{ i\left(\frac{1}{c_1}\frac{\alpha_1}{\alpha_1-1}(c_1 s+d_1)^{\frac{\alpha_1-1}{\alpha_1}} + \theta_{e1}\right)\right\} ds \tag{6}$$

where P_c is the connection point, and θ_{e1} is determined by c_1, d_1, and the direction angle θ_c of the curve at the connection point because the direction angle $\theta(s)$ for a given s is given by

$$\theta_c = \theta(s_c) = \frac{1}{c_0}\frac{\alpha_0}{\alpha_0-1}(c_0 s_c+d_0)^{\frac{\alpha_0-1}{\alpha_0}} + \theta_{e0} \tag{7}$$

for the first segment and

$$\theta_c = \theta(s_c) = \frac{1}{c_1}\frac{\alpha_1}{\alpha_1 - 1}(c_1 s_c + d_1)^{\frac{\alpha_1 - 1}{\alpha_1}} + \theta_{e1} \quad (8)$$

$$\theta_l = \theta(s_l) = \frac{1}{c_1}\frac{\alpha_1}{\alpha_1 - 1}(c_1 s_l + d_1)^{\frac{\alpha_1 - 1}{\alpha_1}} + \theta_{e1} \quad (9)$$

for the second segment.

To generate points on the curve, the equations (5) and (6) must be integrated numerically. Yoshida and Saito used an adaptive Gaussian quadrature method and attained the maximum relative error of 1×10^{-10} to the curve length for a log-aesthetic curve within several milliseconds on a standard PC[YS06]. We use an adaptive quadrature method using Simpson's rule as described in subsection 3.4.

2.3 Connection Conditions

The two segments are connected at the connection point with the continuity of 1) the position, 2) the tangent vector, 3) the radius of curvature, and 4) the derivative of the radius of curvature. Note that the continuity of the radius of curvature and the derivative of the radius of curvature are equivalent to that of curvature and its derivative, respectively, because they are reciprocals of their counterparts. The continuity of the position and the tangent vector is easily guaranteed by specifying suitable values for P_c and θ_{e1}, respectively. The conditions of continuity of the radius of curvature at the connection point are given by:

$$\rho_c = (c_0 s_c + d_0)^{\frac{1}{\alpha_0}} = (c_1 s_c + d_1)^{\frac{1}{\alpha_1}} \quad (10)$$

where ρ_c is the radius of curvature at the connection point. By differentiating $\rho_c^{\alpha_0} = c_0 s_c + d_0$ and $\rho_c^{\alpha_1} = c_1 s_c + d_1$ with respect to the arc length s, the following equations are obtained:

$$\alpha_0 \rho_c^{\alpha_0 - 1}\frac{d\rho_c}{ds} = c_0 \quad (11)$$

$$\alpha_1 \rho_c^{\alpha_1 - 1}\frac{d\rho_c}{ds} = c_1 \quad (12)$$

As the derivative of the radius of curvature $d\rho_c/ds$ is the same for the above two equations, c_1 is given by:

$$c_1 = \frac{\alpha_1}{\alpha_0} c_0 \rho_c^{\alpha_1 - \alpha_0} \quad (13)$$

From Eq.(10), d_1 is given by:

$$d_1 = \rho_c^{\alpha_1} - c_1 s_c \quad (14)$$

α_0 and α_1 are specified by the designer and if the direction angle θ at the connection point and that of the end point are given, the total length of the first segment s_c and that of the whole curve s_l are determined. Hence, if c_0 and d_0 are determined, c_1 and d_1 are uniquely determined because of the conditions of the continuity of the radius of curvature ρ and its derivative $d\rho/ds$ at the connection point P_c.

2.4 Appropriate ranges of c_0 and d_0

From the above discussions, for given start and end points and the tangent directions specified by four control points a compound-rhythm curve is uniquely determined if c_0 and d_0 are determined appropriately. It is necessary to search c_0 and d_0 numerically and by using, for example, Newton's method with derivatives or the downhill simplex method without derivatives (e.g., [PTVF07]). These values are searched to make the end point of the curve equal to the fourth control points. Note that for c_0 and d_0, and also c_1 and d_1 determined by these two values, the radius of curvature must be evaluated as a positive real number in Eq.(4). Therefore, the following inequality equations must be held:

$$c_0 s + d_0 > 0 \quad (0 \le s \le s_c) \quad (15)$$

$$c_1 s + d_1 > 0 \quad (s_c < s \le s_l) \quad (16)$$

Although the length of the first curve segment s_c and the total length of the curve s_l are determined by c_0, d_0, and the direction angles at the start, end, and connection points according to Eq.(2), these values must be positive and appropriately ordered, i.e., $0 < s_c < s_l$.

If one of these constraints is violated, the objective function of the downhill simplex method returns a predefined maximum value or, for example, the distance between the start and end points and naturally the points of the simplex will go away from the violation point.

3 Input of Curve

In this section, we describe how to generate a compound-rhythm log-aesthetic curve from four control points. The designer of the curve inputs two α values for the first and second curve segments as well as the locations of these control points. The α values can be changed by the designer because they are suggested to be related to the impressions of the curve, as mentioned in Section 1.

3.1 Input of Control Points

First, input four control points from P_0 to P_3. By using these points, similar to the cubic Bézier curve, the start point P_0, the end point P_3, the direction angle θ_0 at P_0 and the direction angle θ_1 at P_3 are specified. Furthermore, specify the direction angle θ_c at the connection point by conforming it to the direction angle from P_1 to P_2. Note that if we ignore rigid motion, the number of parameters of a compound-rhythm log-aesthetic curve is essentially four–c_0, d_0, s_c, and s_l– and that of the constraints given is also four–the position of the end point (2 constraints) and the direction angles at the connection and end points (2 constraints). Even if we change the positions of P_1 and P_2, we will obtain the same curve unless the direction from P_1 to P_2 is changed.

For the compound-rhythm curve, its curvature is increasing or decreasing monotonically and it does not have any inflection points. Hence, the polygonal line made of the four control points should be specified as always turning right or left without zigzagging. As the positive curvature is assumed to be obtained if the curve is turning left, in case of the right-turning curve, for example, we create the mirror image of the control points along the line through P_0 and P_1 and generate a curve. Then, we generate the mirror image of the curve as the actual curve specified by the original control points.

3.2 Specification of α_0, α_1

It is necessary for the designer to specify two α values for the first and second curve segments. If necessary, we change the direction of the curve to make its curvature increase monotonically, and then the compound-rhythm curve is classified into two types: a curve whose α changes from positive to negative, and another from negative to positive. The former is called the mountain-type and the latter is the valley-type. Harada et al. noted that valley-type curves were used for several European cars [HMY98]. Generally the designer can specify freely positive or negative values for α_0 and α_1 as required.

However, the log-aesthetic curve segment cannot always be generated from three arbitrarily positioned control points, especially when the absolute value of α is large because the curve segment becomes similar to an arc, as noted previously [YS06]. They also discussed the existence of the solution of the single log-aesthetic curve in detail.

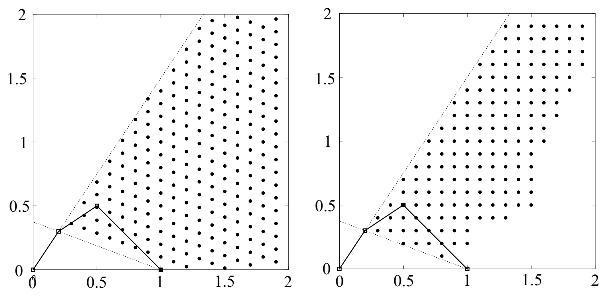
Our algorithm controls the direction at the connection point. Hence the positions of the three control points of each segment are determined and the existence of the solution for each segment of the compound-rhythm log-aesthetic curve is equivalent to that discussed by them. Please refer to their paper for the complete picture of the log-aesthetic curve segment. The compound-rhythm log-aesthetic curve consists of the two log-aesthetic curve segments and its drawable configuration of the four control points depends on those of the log-aesthetic curve segments of the two specified α values.

Figure 1 illustrates the drawable regions of a set of the four control points. The three control points at $(0,0)$, $(0.2,0.3)$, and $(1,0)$ are fixed and the third control point whose initial position is $(0.5,0.5)$ are moved. Each black dot indicates the position of the third control point where a compound-rhythm log-aesthetic curve is drawable. Figure 1(a) and (b) show drawable regions for the curves whose α's are given by $(\alpha_0,\alpha_1) = (-0.1,0.1)$ and $(-2,2)$, respectively. Since the curve cannot have an inflection point except for the start point or the end point, the region where the sequence of the control points becomes zigzagged is undrawable. Generally, as illustrated in [YS06] if the absolute values of α's are larger, the drawable region becomes narrower.

3.3 Search of c_0, d_0

The parameters to determine a compound-rhythm log-aesthetic curve are only two: c_0 and d_0. s_c and s_l are determined by the constraints (directions of the tangent vectors at the connection and end points) from Eqs (7), (8), and (9) if c_0 and d_0 are given.

The values of c_0 and d_0 must be searched numerically. In our implementation, we adopt the downhill simplex method because it does not require the derivatives of the objective function with respect to the parameters. The objective function is the square of the distance between the fourth control point and the end point of the



(a) (α_0,α_1)=(-0.1,0.1) (b) (α_0,α_1)=(-2,2)

Figure 1: Drawable regions for a given set of the four control points. The three control points at $(0,0)$, $(0.2,0.3)$, and $(1,0)$ are fixed and the third control point whose initial position is $(0.5,0.5)$ are moved. Each black dot indicates the position of the third control point where a compound-rhythm log-aesthetic curve is drawable. (a) Drawable region for $(\alpha_0,\alpha_1)=(-0.1,0.1)$. (b) Drawable region for $(\alpha_0,\alpha_1)=(-2,2)$.

curve calculated using the current c_0 and d_0 values. The derivatives with respect to c_0 and d_0 are given by integral forms. Furthermore, as s_c and s_l depend on these values, the derivatives are given by very complex expressions. This makes Newton's method slow.

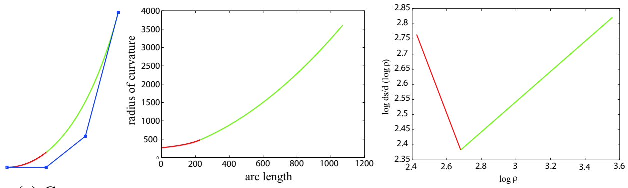
As the initial values of c_0 and d_0, we use those calculated from the curvature values at the start and end points of the cubic Bézier curve defined by the given four control points.

3.4 Curve examples

Figure 2(a) shows an example of the valley-type log-aesthetic curve with compound-rhythm. The curve shown in red is the first segment and that in green is the second segment. The four control points and their connecting lines are shown in blue.

Figure 2(b) shows the graph of the radius of curvature with respect to the arc length. The red and green lines in these graphs (b) as well as (c) correspond to the first and second segments, respectively. Graph (b) indicates that the radius of curvature is continuous between the two segments and it increases monotonically and smoothly.

We specified $\alpha_0=-1.5$, $\alpha_1=0.5$ and the slope of its logarithmic curvature graph changes from negative to positive as shown in Fig. 2(c). Hence, the curve is of valley type. This graph shows that: 1) for each segment of the curve its LCG is given by a straight line segment, i.e., it is a log-aesthetic curve, and 2) as the LCG is continuous, both the derivative of the curvature and the curvature itself are continuous, or C^3 continuity is guaranteed for the whole curve.



(a) Curve (b) Radius of curvature (c) Logarithmic curvature graph

Figure 2: Valley-type compound-rhythm log-aesthetic curve example. (a) Curve with its control points. (b) Graph of arc length v.s. radius of curvature. (c) Logarithmic curvature graph. The red and green lines correspond to the first and second segments, respectively.

Figure 3 shows several examples of the valley- and mountain-type compound-rhythm log-aesthetic curves. α_0 and α_1 are the α values of the first and second segments, respectively. The two types of curve are compared in Fig.3(a). Interestingly, the shape of the valley- and mountain-type curves are similar if we switch their α values from (α_0,α_1) to (α_1,α_0) even though their LCGs are quite different. Further investigations regarding the differences between these two types are necessary. However, but in this paper, we adopted the valley-type curves for car design examples as described in the next section according to the suggestion of [HMY98]. Figure 3(b) shows several examples of valley-type curves. Their shape can vary depending on the α values, and generally if the absolute values of both α_0 and α_1 become larger, the curve is "compressed" because the first and second segments have smaller and larger radii of curvature, respectively.

Figure 3(c) shows several examples of the monotonic-rhythm log-aesthetic curve for comparison with those of compound-rhythm. Note that we cannot simultaneously specify tangent directions at two end points by three control points, and this greatly restricts the applicability of the single log-aesthetic curve segment. Furthermore, the amount of change in the direction angle of the curve input by three control points is theoretically limited within 180° and is at most about 150° to obtain a curve usable for practical design. However, with four control points, it is possible to input a curve the angular change of the direction of which is more

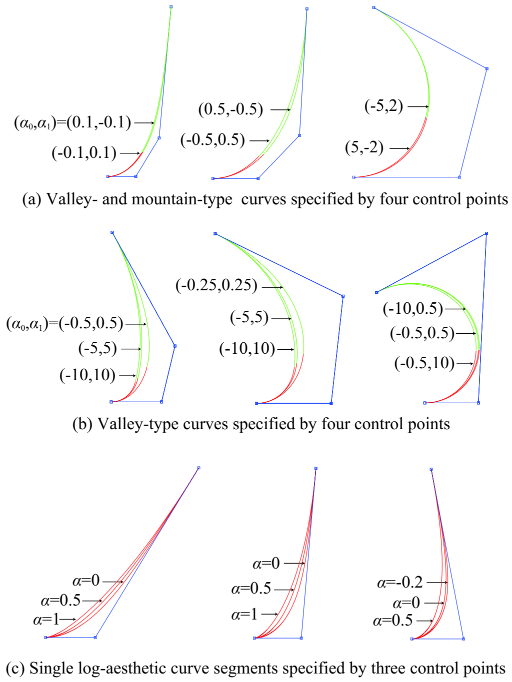


Figure 3: Valley- and mountain-type curves generated by specifying four control points and curves by three control points.

than 180 ° as shown on the right of Fig.3(b).

The processing time spent on the search of c_0 and d_0 in Eq.(5) by the downhill simplex method performed on a PC with a 2.53 GHz Pentium 4 CPU takes about 0.1$\sim$20 ms if a solution exists. We used an adaptive quadrature method using Simpson's rule for numerical integration. We suppressed the maximum error under some epsilon (we used 1.0e-9) multiplied by the curve length. Although numerical integration is necessary to generate a curve, the designer can deform the curve interactively by changing the positions of its control points.

4 Conclusions and Future Work

In this paper, we proposed a method to input a compound-rhythm log-aesthetic curve consisting of two log-aesthetic curve segments using four control points. Using this method, we can generate a curve for α values specified by the designer the logarithmic curvature graph of which is given by a two-segmented polyline.

In future studies, we will examine how to input a curve made up of more than two segments with appropriate continuity levels (G^2 and G^3 continuity) as well as a space curve of log-aesthetic type. We are now extending the log-aesthetic concept into the formulation of a surface. We will develop a styling CAD system for car design transferring the log-aesthetic curve data from the initial design to the manufacturing stage.

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