A General Equation of Aesthetic Curves and Its Self-Affinity

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Abstract

The curve is the most basic design element to determine shapes and silhouettes of industrial products and works for shape designers and it is inevitable for them to make it aesthetic and attractive to improve the total quality of the shape design. If we can find an equation of aesthetic curves, it is expected that the quality of the curve design improves drastically because we can use it as a standard to generate, evaluate, and deform the curves. In this paper, we derive a general equation of aesthetic curves that describes the relationship between its radius of curvature and length inclusively expressing these two curves. Furthermore we show the self-affinity possessed by the curves satisfying the general equation of aesthetic curves.
Keywords: aesthetic curve, general equation of aesthetic curves, self-similarity, self-affinity

1 Introduction

For industrial designers, the curve is one of the most basic design parts that determines shapes and silhouettes of their products and works. It is necessary to make it aesthetically beautiful and attractive to improve the quality of the industrial design.

If we can find an equation of aesthetic curves, it is expected that the quality of the curve design improves drastically because we can use it as a standard to generate, evaluate, and deform the curves.

Therefore in this paper we discuss the properties of two typical aesthetically beautiful curves: the logarithmic spiral and the clothoid curve and we derive a general equation of aesthetic curves that describes the relationship between its radius of curvature and length inclusively expressing these two curves. Furthermore we show the self-affinity possessed by the curves satisfying the general equation of aesthetic curves.

2 General equation of aesthetic curves

Here we will discuss the properties of two typical aesthetic curves: the logarithmic spiral and the clothoid curve. We also discuss the properties of the Archimedean spiral which does not satisfy the general equation of aesthetic curves proposed in this paper as a counterexample to understand the nature of the general equation better.

2.1 Logarithmic spiral

The logarithmic spiral is called the equiangular spiral, or Bernoulli's spiral and is well known as a curve representing the shape of the chambered nautilus. It is closely related to the Golden Section that has been regarded as a source of the beauty since the years of the Greeks and the Romans and is one of the typical beautiful curves as discussed in [5].

2.1.1 Properties of the logarithmic spiral

A logarithmic spiral can be defined in the complex plane by

$$\boldsymbol{C}(t) = e^{(a+ib)t}, \quad (t \geq 0) \tag{1}$$

where i is the imaginary unit and a and b are constants. Since its radius of curvature $\rho(t)$ and the arc length $s(t)$ are given by

$$\rho(t) = \frac{1}{b}\sqrt{a^2+b^2}e^{at}, \quad s(t) = \sqrt{a^2+b^2}(e^{at}-1), \tag{2}$$

the following equation is satisfied.

$$\rho(t) = c_0 s(t) + c_1 \tag{3}$$

Figure 1 shows an example of the logarithmic spiral.

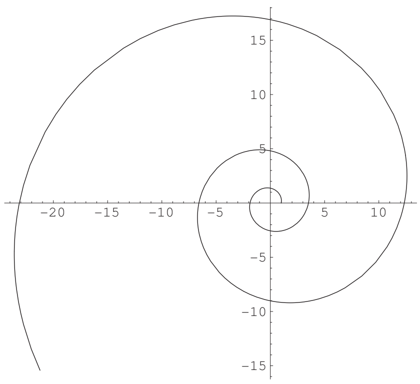

Figure 1: Logarithmic spiral (a=0.2, b=1)

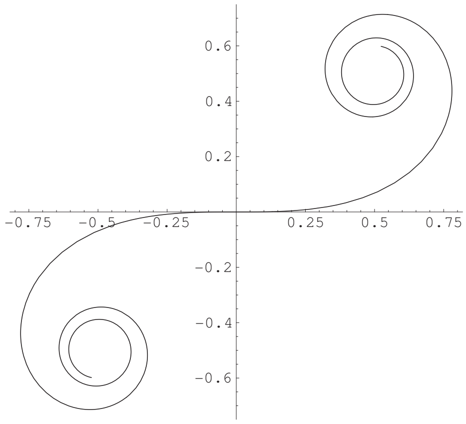

Figure 2: Clothoid curve (a=1)

2.1.2 Self-similarity

The self-similarity is a characteristic property of the fractal geometry and it becomes a similar shape to the original after scaling it like a saw-toothed coastline[9]. We will show that the logarithmic spiral has the self-similarity below.

A logarithmic spiral is given by Eq.(1) and we cut the head portion of the curve and define a new curve $\boldsymbol{D}(t)$ for $t \geq 1$ of $\boldsymbol{C}(t)$ as follows:

$$\boldsymbol{D}(t) = \boldsymbol{C}(t+1) = e^a e^{ib} \boldsymbol{C}(t) \tag{4}$$

As we see from the above equation, the curve $\boldsymbol{D}(t)$ can be obtained by scaling $\boldsymbol{C}(t)$ by the factor e^a and rotating it by the angle b about the origin. Therefore since the original curve is recovered by scaling the curve whose head portion is cut off, the logarithmic spiral has the self-similarity. Here we removed the head portion where $t < 1$, but it is rather obvious to be able to argue similarly by cutting an arbitrary head portion.

2.2 Clothoid curve

The clothoid curve is also called Cornu's spiral and is regarded one of the beautiful curves (for example, see [6]).

2.2.1 Properties of the clothoid curve

One of the main properties of the clothoid curve is that its curvature increases in proportion to its arc length. Figure 2 shows an example of the clothoid curve.

A clothoid curve can be defined in the complex plane by

$$\boldsymbol{C}(t) = \int_0^t e^{iat^2} dt \tag{5}$$

where a is a positive constant. The first derivative of $\boldsymbol{C}(t)$ is

$$\frac{d\boldsymbol{C}(t)}{dt} = e^{iat^2} \tag{6}$$

and its absolute value is always equal to 1. Hence the parameter t is the same as the arc length $s(t)$(for example, see [8]). Then the curvature is given by the absolute value of the second derivative

$$\kappa(t) = \left|\frac{d^2\boldsymbol{C}(t)}{dt^2}\right| = |(2iat)e^{iat^2}| = 2at \tag{7}$$

If we consider cutting off the head portion of the curve again, the radius of curvature$\rho(t) = 1/\kappa(t)$ is given by

$$\rho(t)^{-1} = c_0\, s(t) + c_1 \tag{8}$$

2.2.2 Self-affinity

Although the self-similarity can be found ubiquitously in the natural world, not so many phenomena of the self-affinity are known. Some kind of the Brownian motion has such a self-affinity that by doubling the scale of the time and scaling its amplitude by $\sqrt{2}$, it shows the self-similarity[10]. That means the self-similarity by scaling in the different coordinate axes by different values is called the self-affinity. We will discuss the self-affinity possessed by the clothoid curve below.

Similar to the logarithmic spiral case, we consider cutting off the head portion of the curve and we define the curve $\boldsymbol{D}(t)$ whose parameter $t \geq 1$ as follows:

$$\begin{aligned} \boldsymbol{D}(t) &= \boldsymbol{C}(t+1) \\ &= \int_0^1 e^{iat^2} dt + \int_1^t e^{iat^2} dt \\ &= \boldsymbol{P}_0 + \int_1^t e^{iat^2} dt \end{aligned} \tag{9}$$

where the start point of $\boldsymbol{D}(t)$ is given by $\boldsymbol{P}_0 = \int_0^1 e^{iat^2} dt$. Since its shape is invariant under reparametrization of $s(t)$ by an arbitrary monotonously increasing function of t, reparametrize $s(t) = c_1(e^{\beta t} - 1)/c_0$ with assuming that β is a positive constant. Then the arc length $s_D(t)$ of $\boldsymbol{D}(t)$ is given by

$$\begin{aligned} s_D(t) &= s(t+1) - s(1) \\ &= e^{\beta} s(t) \end{aligned} \tag{10}$$

Therefore the arc length of $\boldsymbol{D}(t)$ is obtained by scaling that of the original by e^{β}.

From Eq.(8) the inverse of the radius of curvature $\rho_D(t)$ of $\boldsymbol{D}(t)$ is

$$\begin{aligned} \rho_D(t)^{-1} &= \rho(t+1)^{-1} \\ &= e^{\beta}(c_0 s(t) + c_1) \end{aligned} \tag{11}$$

Hence

$$\rho_D(t) = e^{-\beta} \rho(t) \tag{12}$$

This means that the radius of curvature of the curve without the head portion is given by scaling that of the original curve by $e^{\frac{\beta}{\alpha}}$.

The clothoid curve without the head portion is identical with that generated by scaling the radius of curvature of the original curve in the principal normal direction by $e^{\beta/\alpha}$ and its arc length in the tangent direction by e^{β}. This means that at an arbitrary point on the curve in the two different orthogonal directions, the principal normal and the tangent by scaling the cut curve by the different factors, the original curve can be obtained. This property can be called the self-affinity.

We will discuss the self-affinity of the curves that satisfy the general equation of aesthetic curve in Section 5 in more detail.

2.3 General equation of aesthetic curves

We can derive the following general equation including both Eqs.(3) and (8):

$$\rho(t)^{\alpha} = c_0 s(t) + c_1 \tag{13}$$

where $\alpha \neq 0$ is a constant. When $\alpha = 1$ and $\alpha = -1$, we obtain Eqs.(3) and (8) respectively.

As Equation (13) can express the two typical aesthetic curves: the logarithmic spiral and the clothoid curves and it has desirable properties discussed in the following sections, we call it a general equation of aesthetic curves in this paper.

2.4 Counterexample

We discuss the properties of the Archimedean spiral whose logarithmic curvature histogram can not be approximated by a straight line properly as a counterexample of aesthetic curves. We will mention the logarithmic curvature histogram in the next section.

The Archimedean spiral is also called the uniform spiral and is a spiral whose radius increases in proportion to the angle to the x-axis as shown in Fig.3. In the complex plane, its general expression is given by

$$\boldsymbol{C}(t) = ate^{ibt}, \quad (t \geq 0) \tag{14}$$

where a and b are constants.

The definition of the Archimedean spiral is simply given by Eq.(14) and has a geometrically regular property that the intersection intervals on the x and y axes are constant However, the main usage of the Archimedean spiral is for the design of machines such as water pumps and it is not so frequently used for aesthetic design purposes.

3 Logarithmic curvature histogram

Harada et al.[1, 2] insisted that natural aesthetic curves like birds' eggs and butterflies' wings as well as artificial ones like Japanese swords and key lines of automobiles have such a property that their logarithmic curvature histograms(LCHs)

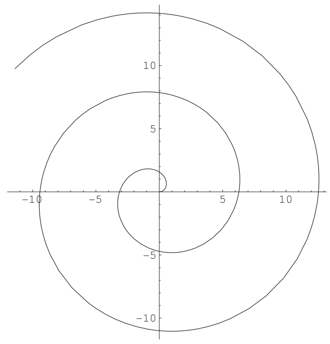

Figure 3: Archimedean spiral (a=1,b=1)

can be approximated by straight lines and there is a strong correlation between the slopes of the lines and the impressions of the curves.

Since the vertical value of the LCH is given by $\log |ds/d(\log \rho)|$[3] and both s and ρ are functions of the parameter t,

$$\begin{aligned}\log\left|\frac{ds}{d(\log \rho)}\right| &= \log\left|\frac{\frac{ds}{dt}}{\frac{d(\log \rho)}{dt}}\right| = \log(\rho\left|\frac{\frac{ds}{dt}}{\frac{d\rho}{dt}}\right|) \\ &= \log \rho + \log s_d - \log\left|\frac{d\rho}{dt}\right| \end{aligned} \tag{15}$$

If we assume $t = s$, then $ds/dt = s_d = 1$. The above equation can be transformed using (13) to

$$\begin{aligned}\log\left|\frac{ds}{d(\log \rho)}\right| &= \log \rho - \log(\left|\frac{c_0}{\alpha}\right|\rho^{1-\alpha}) \\ &= \alpha \log \rho + C \end{aligned} \tag{16}$$

where $C = \log |\alpha| - \log |c_0|$. Therefore the LCH of the curve satisfying the general equation of aesthetic curves is strictly given by a straight line whose slope is equal to α.

4 Parametric expressions

Equation (13) describes only the relation between the radius of curvature and arc length of the curve and it is not suitable to draw it or analyze its properties. In this section, we derive two parametric expressions of the general equation of aesthetic curves given by Eq.(13). One is derived directly from the general equation and the other is done by applying the fine tuning method[7] to the clothoid curve.

4.1 Parametric expression of the general aesthetic curve

We assume that a curve $\boldsymbol{C}(s)$ satisfies Eq.(13). Then

$$\rho(s) = (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{17}$$

As s is the arc length, $|s_d| = 1$(refer to, for example, [8]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{18}$$

Since $\rho(s) = 1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{19}$$

Hence

$$\theta = \frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0} + c_2 \tag{20}$$

If the start point of the curve is given by $\boldsymbol{P}_0 = \boldsymbol{C}(0)$,

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i\frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0}} ds \tag{21}$$

The above expression can be regarded as an extension of the clothoid curve whose power of e in its definition is changed from 2 to $\alpha + 1$ and its LCH line's slope can be specified to be equal to any value except for 0.

4.2 Another parametric expression

Here we will apply the fine tuning method developed by Miura et al.[7] to the clothoid curve and extend its representation. The fine tuning method can scale curvature at a point on curves and surfaces to an arbitrary value. In the curve case, for a given curve $\boldsymbol{C}(t)$, by using a scalar function $g(t) > 0$ and define a new curve as follows:

$$\boldsymbol{C}'(t) = \boldsymbol{P}_0 + \int_0^t g(t)\frac{d\boldsymbol{C}(t)}{dt} dt \tag{22}$$

Namely differentiate the original curve, scale the first derivative by multiplying a scale function and change the value of curvature arbitrarily. The clothoid curve applied by the fine tuning(Fine Tuned Clothoid : FTC) is defined by the following expression in the complex plane:

$$\boldsymbol{C}(t) = \int_0^t g(t) e^{iat^2} dt \tag{23}$$

where i is the imaginary unit, a is a constant and $g(t)$ is a scale function whose value is always positive.

By using the radius of curvature ρ_c of the clothoid curve, we define $g(t) = (1/2at)^\beta$ If we assume β can be positive or negative values, $g(t)$ is equivalent to be the $-\beta$-th power of t except for the constant coefficient. The analysis results yield

$$\log \Delta s \quad = \quad \frac{\beta - 1}{\beta + 1} \log \rho + C \tag{24}$$

where $C = -\log(\beta+1) - \log 2 - \log a + \log c$. Hence the LCH graph is given by a straight line whose slope is $(\beta-1)/(\beta+1)$ and the slope α can be an arbitrary value except for 1[1]. Figure 4 shows several FTC curves whose LCH lines' slopes are given by α. The curve whose α is equal to -1 is a clothoid curve.

The FTC curve which has 1 for its LCH line slope can be obtained with $g(t) = c_0 t e^{c_1 t^2}$ by solving a differential equation $\Delta s/\rho = const$ where c_0 and c_1 are constants. In this case, we can perform the integration explicitly and it turns out to be a logarithmic spiral expressed by

$$\boldsymbol{C}(t) = e^{ic_2} \int_0^t c_0 t e^{c_1 t^2} e^{iat^2} dt = \frac{c_0}{2(c_1 + ia)} e^{ic_2} e^{(c_1+ia)t^2} \tag{25}$$

where c_2 is a integration constant.

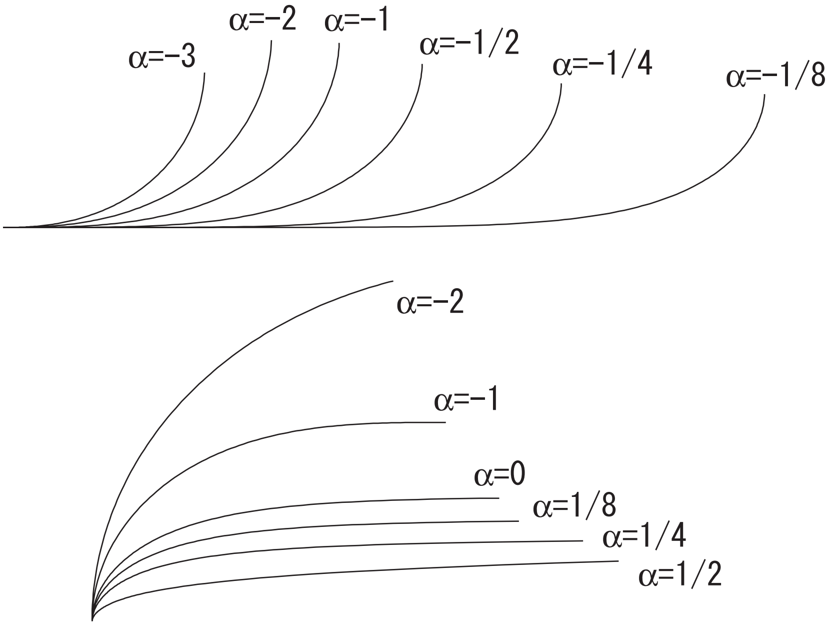

Figure 4: Curves whose LCH graphs are given by α-sloped straight lines

5 Self-Affinity

Harada et al.[1] addressed that the curve whose logarithmic curvature histogram was expressed by a straight line had a self-affinity, but his proof was not mathematically strict. His statement that "the property is called the self-affinity of the curve that the curve obtained by cutting the original curve at two positions and applying such an affine matrix that scales by two different scaling factors in the two orthogonal directions becomes identical to the original curve" is misleading. It might be interpreted that there is a 2×2 matrix depending only on the cutting positions. However it is trivial that there is not such a matrix for a

[1] If β is equal to -1, the curve becomes a circle

clothoid curve[2]. It means we need a new definition of the self-affinity for aesthetic curves possessed by those who satisfy the general equations of aesthetic curves.

5.1 Self-affinity of aesthetic curves

We have already shown that the clothoid curve has a self-affinity property. Here we will discuss the self-affinity possessed by the curves that satisfy the general equation of aesthetic curves below.

Assume that a curve satisfies the general equation of aesthetic curves expressed by Eq.(13). Then for a given α,

$$\rho(t)^\alpha = c_0 s(t) + c_1 \tag{26}$$

As even if $s(t)$ is reparametrized by an arbitrary monotonously increasing function, the shape remains the same, we reparametrize the curve by $s(t) = c_1(e^{\beta t} - 1)/c_0$. Then

$$\rho(t) = c_1^{\frac{1}{\alpha}} e^{\frac{\beta}{\alpha} t} \tag{27}$$

Similar to the previous subsection, we get a curve $\boldsymbol{D}(t)$ by cutting the head portion of the curve by substitute t with $t+1$ and obtain its radius of curvature $\rho_D(t)$

$$\rho_D(t) = \rho(t+1) = c_1^{\frac{1}{\alpha}} e^{\frac{\beta}{\alpha}} e^{\frac{\beta}{\alpha} t} \tag{28}$$

Hence the radius of curvature of the curve without the head portion is given by scaling that of the original curve by $e^{\frac{\beta}{\alpha}}$.

The arc length of the curve $s_D(t)$ is

$$s_D(t) = s(t+1) - s(t) = \frac{c_1}{c_0} e^{\beta}(e^{\beta t} - 1) \tag{29}$$

Therefore the arc length of $\boldsymbol{D}(t)$ is obtained by scaling that of the original by e^{β}.

5.2 Self-affinity of the curve

In summary, the curve without the head portion is identical with that generated by scaling the radius of curvature of the original curve in the principal normal direction by $e^{\beta/\alpha}$ and its arc length in the tangent direction by e^{β}, or in the Frenet frame. This means that at an arbitrary point on the curve in the two different orthogonal directions, the principal normal and the tangent by scaling the cut curve by the different factors, the original curve can be obtained. We define this kind of the self-affinity as that of aesthetic curves.

The self-affinity of the Brownian motion introduced in this subsection is for a fixed coordinate system made by the time and the amplitude axes, but the self-affinity of aesthetic curves is for a moving coordinate made by the principal normal and tangent directions along the curve. Although the matrix used for the affine transformation is the same in the moving coordinate system, no affine matrix exists for a fixed coordinate system.

[2]If we apply an affine matrix to a multi-looped clothoid curve, the curve will warp and not be another clothoid curve.

6 Conclusion

Based on the discussions about the two typical aesthetic curves: the logarithmic spiral and the clothoid curve, we have derived a general equation of aesthetic curves describing the relationship between the radius of curvature and the arc length of the curve. We have shown a curve satisfying the general equation has such a property that its LCH graph is given by a straight line. We have found two types of parametric expressions for the general aesthetic curve: the extended clothoid and fine tuned clothoid curves. We have also shown that the curve satisfying the general equation of aesthetic curves has some kind of a self-affinity and defined it as the self-affinity of aesthetic curves.

For future work, we are planning an automatic classification of curves: 1) determine the rhythm to be simple(monotonic) or complex(consisting of plural rhythms), 2) calculate the slope of the line approximating the LCH graph. We think there are a lot of possibilities to use the general aesthetic equations to many applications in the fields of computer aided geometric design. For example, we may be able to apply the equations to deform curves to change their impressions, say, from sharp to stable. Another example is smoothing for reverse engineering. Even if only noisy data of curves are available, we may be able to use the equations as kinds of rulers to smooth out the data and yield aesthetically high quality curves. We will develop a CAD system using the equations.

References

[1] T. Harada, N. Mori, K. Sugiyama, "Curves' Physical Characteristics and Self-affine properties," , Vol.42, No.3, pp.30-40, 1995(Japanese).

[2] T. Harada, "Study of Quantitative Analysis of the Characteristics of a Curve," Forma, Vol. 12 (No. 1), pp. 55-63, 1997.

[3] I. Kanaya, Y. Nakano, K. Sato, "Simulated designer's eyes -classification of aesthetic surfaces-," Proc. VSMM2003, pp. 289-296, 2003.

[4] M. Higashi, K. Kaneko, M. Hosaka, "Generation of High-Quality Curve and Surface with Smoothly Varying Curvature, Proc. of Eurographics'88, 79-86, 1988.

[5] M. Livio, The Golden Ratio, Broadway Books, 2002.

[6] T. Takanashi, Aesthetic Design Methodology, David press, 2002.

[7] K.T. Miura, F. Cheng, L. Wang, "Fine Tuning: Curve and Surface Deformation by Scaling Derivatives," Proc. Pacific Graphics 2001, pp.150-159, 2001.

[8] G. Farin, Curves and Surfaces for CAGD, 5th Ed., Morgan Kaufmann, 2001.

[9] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman and Company, 1983.

[10] T. Takagi, Mathematics on Shapes, Asakura Shoten, 1992(Japanese).