Variational Formulation of the Log-Aesthetic Curve

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ABSTRACT

The log-aesthetic curves include the logarithmic (equiangular) spiral, clothoid, and involute curves. Although most of them are expressed only by an integral form of the tangent vector, it is possible to interactively generate and deform them and they are expected to be utilized for practical use of industrial and graphical design. We reformulate the LA curve with variational principle in order to analyze its properties.

Categories and Subject Descriptors

I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling|curve, surface, solid and object representations

General Terms

Theory

Keywords

log-aesthetic curve, variational principle, radius of curvature, differential equation

1. INTRODUCTION

The log-aesthetic curves include the logarithmic (equiangular) curve (the slope of the LCG: logarithmic curvature graph $\alpha = 1$), the clothoid curve ($\alpha = -1$), the circle involute ($\alpha = 2$) and Nielsen's spiral ($\alpha = 0$). Recently the generalized Cornu spiral[6] has been reported to include several log-aesthetic curves since its curvature profile is given by a rational linear function and so its LCG gradient is given by a straight line function[4]. It is possible to generate and deform the log-aesthetic curve in real time even if they are expressed by integral forms using their unit tangent vectors as integrands ($\alpha \neq 1, 2$) and they are expected to be used in practical applications [1, 20].

Furthermore recently Ziatdinov et al.[14] showed that the log-aesthetic curve can be parametrically expressed in terms of incomplete gamma functions, which gives an exact analytic representation of a curve segment for any real value of α and the computation time for generating a log-aesthetic curve segment using the incomplete gamma functions is about 10 times faster than using direct numerical integration.

The discrete log-aesthetic filter based on the formulation of the log-aesthetic curve has successfully been introduced not to impose strong constraints on the designer's activity, to let him/her design freely and to embed the properties of the log-aesthetic curves for complicated curves with both increasing and decreasing curvature[8]. In this paper we define the log-aesthetic curve based on variational principle in order to clarify its properties.

The rest of the paper is organized as follows. Section 2 describes related work and sections 3 explains the formulation of the log-aesthetic curve based on variational principle and discusses a derivation of a non-linear ordinary differential equation which is satisfied by the LA curve. Furthermore it justifies the correctness of the variational formulation by solving the ordinary differential equation derived from the variational formulation. Finally, we conclude the paper in section 4 with a discussion of future work.

2. RELATED WORK

In this section, we discuss related researches on the log-aesthetic curve, curvature based energy functionals for fair surfaces, and discrete filters.

2.1 Log-aesthetic Curve

Aesthetic curves were proposed by Harada et al.[5] as such curves whose logarithmic distribution diagram of curvature (LDDC) is approximated by a straight line. Miura et al. [11, 7] derived analytical solutions of the curves whose logarithmic curvature graph (LCG): an analytical version of the LDDC[5] are strictly given by a straight line and proposed these lines as general equations of aesthetic curves. Furthermore, Yoshida and Saito[12] analyzed the properties of the curves expressed by the general equations and developed a new method to interactively generate a curve by specifying two end points and the tangent vectors there with three control points as well as : the slope of the straight line of the LCG. In this research, we call the curves expressed by the general equations of aesthetic curves the log-aesthetic curves.

The problems of the connection of plural log-aesthetic seg-

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ments was dealt by Miura et al.[9] and an input method of the compound-rhythm log-aesthetic curve which consists of two log-aesthetic curve segments connected with C^3 continuity was proposed by Agari[1]. Furthermore an extension of the planar log-aesthetic curve into space: the log-aesthetic space curve was proposed by Miura et al.[10] and it was classified by Yoshida and Saito[13]. This section discusses several important properties of log-aesthetic curves. Note that an aesthetic curve is a curve whose logarithmic curvature graph is given by a straight line.

2.1.1 General Equations of Aesthetic Curves

For a given curve, we assume the arc length of the curve and the radius of curvature are denoted by s and ρ, respectively. The horizontal axis of the logarithmic curvature graph measures $\log \rho$ and the vertical axis measures $\log(ds/d(\log \rho)) = \log(\rho ds/d\rho)$. If the LCG is given by a straight line, there exists a constant such that the following equation is satisfied:

$$\log(\rho \frac{ds}{d\rho}) = \alpha \log \rho + c \tag{1}$$

where c is a constant. The above equation is called the fundamental equation of aesthetic curves[11]. Rewriting Eq.(1), we obtain:

$$\frac{1}{\rho^{\alpha-1}} \frac{ds}{d\rho} = e^c = c' \tag{2}$$

Hence there is some constant c' such that:

$$\rho^{\alpha-1} \frac{d\rho}{ds} = c' \tag{3}$$

From the above equation, when $\alpha \neq 0$, for some constants c_0 and c_1 the first general equation of aesthetic curves

$$\rho^{\alpha} = c_0 s + c_1 \tag{4}$$

is obtained. If $\alpha = 0$, we obtain the second general equation of aesthetic curves

$$\rho = c_0 e^{c_1 s} \tag{5}$$

The curve which satisfies Eq.(4) or Eq.(5) is called the log-aesthetic curve.

2.1.2 Parametric Expressions of the LA Curves

In this subsection, we will show parametric expressions of the log-aesthetic curves. We assume that a curve satisfies Eq.(4). Then

$$\rho(s) = (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{6}$$

As s is the arc length, (refer to, for example, [2, 3]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{7}$$

Since $\rho = 1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{8}$$

If $\alpha \neq 1$,

$$\theta = \frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0} + c_2 \tag{9}$$

For a given curve $\boldsymbol{C}(s)$, if the start point of the curve is given by $\boldsymbol{P}_0 = \boldsymbol{C}(0)$,

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i\frac{\alpha(c_0 u + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0}} du \tag{10}$$

where i is the imaginary unit. For the second general equation of aesthetic curves expressed by Eq.(5),

$$\frac{d\theta}{ds} = \frac{1}{c_0} e^{-c_1 s} \tag{11}$$

$$\theta = -\frac{1}{c_0 c_1} e^{c_1 s} + c_2 \tag{12}$$

Therefore the curve is given by

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{-\frac{i}{c_0 c_1} e^{-c_1 u}} du \tag{13}$$

3. VARIATIONAL FORMULATION

In this section, at first we discuss about the variational principle with a simple example and explain how to formulate the log-aesthetic curve, especially about the functional which the log-aesthetic curve minimizes.

3.1 Variational Principle

The variational analysis deals with a problem where an objective functional in an integral form should be minimized or maximized. For examples,

$$J = \int_{x_1}^{x_2} f(y, y_x, x) dx \tag{14}$$

where y is a function of x and y_x is a derivative of y with respect to x. y is unknown. The condition that J has a stationary value is given by the following partial differential equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0 \tag{15}$$

This equation is called the Euler equation. If $f = f(y, y_x)$, i.e. f is given explicitly without x, the above equation means that

$$f - y_x \frac{\partial f}{\partial y_x} = c \tag{16}$$

where c is a constant.

The simplest example of the variational problem is to minimize the distance between two given points in the $x - y$ plane. An infinitesimal element of the distance is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y_x^2} dx \tag{17}$$

and the distance J is given by

$$J = \int_{x_1, y_1}^{x_2, y_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y_x^2} dx \tag{18}$$

Hence $f(y, y_x, x) = \sqrt{(1 + y_x^2)}$ and is given explicitly without x. By Eq.(16) we obtain

$$\frac{1}{\sqrt{1 + y_x^2}} = c \tag{19}$$

Therefore there exists a constant a such that $y_x = a$. It yields

$$y = ax + b \tag{20}$$

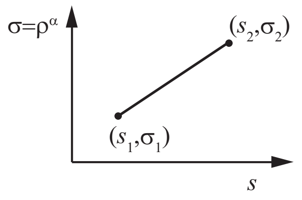

Figure 1: A straight line segment in the $s-\sigma$ plane connecting two points (s_1, σ_1) and (s_2, σ_2).

where b is a constant as well as a. These constants are determined by making the line pass through the given two points (x_1, y_1) and (x_2, y_2).

3.2 Variational Formulation of the LA Curve

In Eq.(4) if we substitute ρ^α with σ, then the equation is given by

$$\sigma = c_0 s + c_1 \tag{21}$$

The above equation means that the log-aesthetic curve is given by a straight line in the $s-\sigma$ plane where the horizontal and vertical axes are the arc length s and σ, respectively to connect two given points (s_1, σ_1) and (s_2, σ_2) as shown in Fig.1. In this case the following objective functional J_{LAC} is minimized.

$$J_{LAC} = \int_{s_1}^{s_2} \sqrt{1+\sigma_s^2}ds = \int_{s_1}^{s_2} \sqrt{1+\alpha^2\rho^{2\alpha-2}\rho_s^2}ds \tag{22}$$

3.2.1 $\alpha = 1$

If $\alpha = 1$, the curve is a logarithmic (equiangular) curve. For the curve f in Eq.(22)is given by

$$f = \sqrt{1+\rho_s^2} \tag{23}$$

and it means that in the plane whose horizontal and vertical axes are the arc length s and the radius of curvature ρ the shortest curve connecting two points (s_1, ρ_1) and (s_2, ρ_2) is a logarithmic spiral.

3.2.2 $\alpha = -1$

If $\alpha = -1$, the curve is a clothoid curve. For the curve f in Eq.(22) is given by

$$f = \sqrt{1+\kappa_s^2} \tag{24}$$

and it means that in the plane whose horizontal and vertical axes are the arc length s and the curvature $\kappa = 1/\rho$ the shortest curve connecting two points (s_1, κ_1) and (s_2, κ_2) is a clothoid curve.

3.3 Generalization of the Parameter

We assume the curve is defined by a general parameter instead of the arc length s. Then

$$\begin{aligned} ds &= \sqrt{x_t^2+y_t^2}dt \\ \rho_s &= \frac{\frac{d\rho}{dt}}{\frac{ds}{dt}} \\ &= \frac{\rho_t}{\sqrt{x_t^2+y_t^2}} \end{aligned} \tag{25}$$

Eq.(22) is given by

$$\begin{aligned} J_{LAC} &= \int_{t_1}^{t_2} \sqrt{1+\alpha^2\rho^{2\alpha-1}\frac{\rho_t^2}{x_t^2+y_t^2}}\sqrt{x_t^2+y_t^2}dt \\ &= \int_{t_1}^{t_2} \sqrt{x_t^2+y_t^2+\alpha^2\rho^{2\alpha-2}\rho_t^2}dt \end{aligned} \tag{26}$$

Therefore

$$f(t) = \sqrt{x_t^2+y_t^2+\alpha^2\rho^{2\alpha-2}\rho_t^2} \tag{27}$$

3.4 Non-linear Differential Equation

Here we will calculate Eq.(15) for Eq.(22).

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{1}{2}(1+\alpha^2\rho^{2\alpha-2}\rho_s^2)^{-\frac{1}{2}}\alpha^2(2\alpha-2)\rho^{2\alpha-3}\rho_s^2 \\ \frac{\partial f}{\partial \rho_s} &= (1+\alpha^2\rho^{2\alpha-2}\rho_s^2)^{-\frac{1}{2}}\alpha^2\rho^{2\alpha-2}\rho_s \end{aligned} \tag{28}$$

Furthermore

$$\begin{aligned} \frac{d}{ds}\frac{\partial f}{\partial \rho_s} &= -\frac{1}{2}(1+\alpha^2\rho^{2\alpha-2}\rho_s^2)^{-\frac{3}{2}}(\alpha^2(2\alpha-2)\rho^{2\alpha-3}\rho_s^3 \\ &\quad +2\alpha^2\rho^{2\alpha-2}\rho_s\rho_{ss})\alpha^2\rho^{2\alpha-2}\rho_s \\ &\quad +(1+\alpha^2\rho^{2\alpha-2}\rho_s^2)^{-\frac{1}{2}}(\alpha^2(2\alpha-2)\rho^{2\alpha-3}\rho_s^2 \\ &\quad +\alpha^2\rho^{2\alpha-2}\rho_{ss}) \end{aligned} \tag{29}$$

Hence Eq.(15) is given by

$$\frac{\partial f}{\partial \rho} - \frac{d}{ds}\frac{\partial f}{\partial \rho_s} = -\frac{\alpha^2\rho^{2\alpha-3}((\alpha-1)\rho_s^2+\rho\rho_{ss})}{(1+\alpha^2\rho^{2\alpha-2}\rho_s^2)^{\frac{3}{2}}} = 0 \tag{30}$$

Therefore

$$(\alpha-1)\rho_s^2+\rho\rho_{ss} = 0 \tag{31}$$

On the other hand, since for the log-aesthetic curve, ρ^α is a linear function of s,

$$\frac{d^2\rho^\alpha}{ds^2} = \alpha\rho^{\alpha-2}((\alpha-1)\rho_s^2+\rho\rho_{ss}) = 0 \tag{32}$$

Hence

$$(\alpha-1)\rho_s^2+\rho\rho_{ss} = 0 \tag{33}$$

The above equation is identical to Eq.(31).

Since $\rho = 1/\kappa$, $\rho_s = -\kappa_s/\kappa^2$, and $\rho_{ss} = -(\kappa_{ss}\kappa + 2\kappa_s)/\kappa^3$, Eq.(31) can be rewritten as follows:

$$\kappa_{ss} = (\alpha+1)\frac{\kappa_s^2}{\kappa} \tag{34}$$

3.5 Solutions of $(\alpha-1)\rho_s^2+\rho\rho_{ss} = 0$

We will solve Eq.(31). Although this equation is a non-linear second order ordinary differential equation with an independent variable s, it does not include s explicitly. We exchange the roles of ρ and s and let ρ be an independent variable and s be a dependent one. Since $\rho_s = 1/s_\rho$, $\rho_{ss} = -s_{\rho\rho}/s_\rho^3$, Eq.(31) is rewritten as

$$\begin{aligned} \frac{\alpha-1}{s_\rho^2} - \rho\frac{s_{\rho\rho}}{s_\rho^3} &= 0 \\ (\alpha-1)s_\rho - \rho s_{\rho\rho} &= 0 \end{aligned} \tag{35}$$

Let $t = s_\rho$, then

$$t_\rho = (\alpha - 1)\frac{t}{\rho} \tag{36}$$

Note that the above equation is one of Abel's ordinary differential equations ($y'(x) = f(x) + g(x)y(x) + h(x)y(x)^2 + k(x)y(x)^3$). Hence

$$t = c_0 r^{\alpha-1} \tag{37}$$

If $\alpha \neq 0$,

$$s = \frac{c0}{\alpha}\rho^\alpha + c_1 \tag{38}$$

If $\alpha = 0$,

$$s = c_0 \log r + c_1 \tag{39}$$

The above two equations are the same as those of the log-aesthetic curve.

4. SPACE CURVE

The log-aesthetic space curve was proposed by Miura et al.[10] and they used the Frenet-Serret formula (for example, see [3]).

4.1 The Frenet-Serret formula

For a space curve $\boldsymbol{C}(s)$ parameterized by s, let its unit tangent vector to be $\boldsymbol{t}$, unit principal normal vector $\boldsymbol{n}$, and unit binormal vector $\boldsymbol{b}$. These vectors are related by the Frenet-Serret formula as follows:

$$\begin{aligned} \frac{d\boldsymbol{C}(s)}{ds} &= \boldsymbol{t}, \quad \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}, \\ \frac{d\boldsymbol{n}}{ds} &= -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \quad \frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{n} \end{aligned} \tag{40}$$

where κ and τ are the curvature and torsion, respectively.

The plane curve has a constant binormal vector and its torsion remains 0. But we have to consider its change for the space curve. Hence first, we define self-affinity of the space curve and next we define the aesthetic space curve as the curve who has self-affinity.

Similar to self-affinity of the plane curve, we define self-affinity of the space curve as follows. For a curve generated by removing arbitrary head portion of the original curve, by scaling it with different factors in its tangent, principal normal and binormal directions on every point on the curve, if the original curve is obtained, then the curve has self-affinity.

Since the curvature and torsion, or their reciprocals, i.e., the radius of curvature and radius of torsion can be independently specified, for the radius of torsion $\mu = 1/\tau$, we assume that an equation similar to Eq.(1) is satisfied as follows:

$$\log(\mu\frac{ds}{d\mu}) = \beta \log \mu + d \tag{41}$$

where β and d are constants. Then

$$\mu^{\beta-1}\frac{d\mu}{ds} = d_0 \tag{42}$$

where d_0 is a constant. As Miura et al.[10] indicated, similar to the argument that shows that a sufficient and necessary condition to have self-affinity of the plane curve is expressed by Eq.(3), it can be shown that a sufficient and necessary condition to have self-affinity of the space curve is expressed by Eqs.(3) and (42).

From Eq.(42), when $\beta \neq 0$, for two constants d_0 and d_1 the first general equation of aesthetic curves on the radius of torsion

$$\mu^\beta = d_0 s + d_1 \tag{43}$$

is obtained. If $\beta = 0$, we obtain the second general equation of aesthetic curves on the radius of torsion

$$\mu = d_0 e^{d_1 s} \tag{44}$$

The Frenet-Serret formula can be considered to be simultanious differential equations and an example calculated by their numerial integration is shown in Fig.2. The top and bottom figures shows the same five curves from different viewpoints and the curve drawn at the bottoms is identical to a logarithmic spiral whose torsion is always 0 and radius of curvature is given by a linear function of the arc length. The other curves have the same start point and radius of curvature as the logarithmic spiral and their torsion is given by a linear function of the arc length with $\beta = 1$. The upper curves have smaller coefficient of the linear function for the arc length (larger torsion). For each curve, at the start and end points, and two points on the curve, we draw the tangent, principal normal and binormal vectors of the moving frame (Frenet frame) as short slim cyliders.

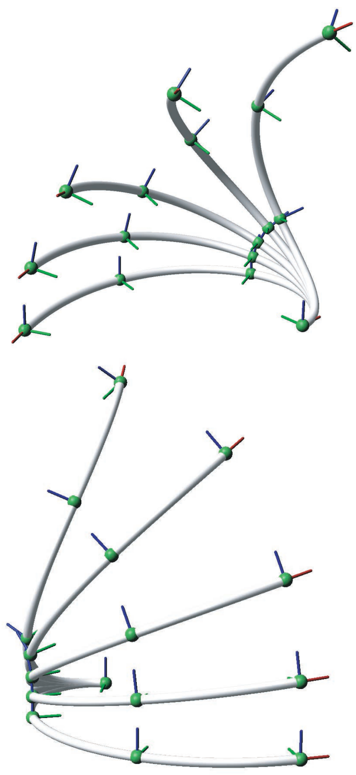

Figure 2: Examples of the aesthetic space curve

4.2 Variational Formulation

In Eq.(43) if we substitute μ^β with ϕ, then the equation is given by

$$\phi = d_0 s + d_1 \quad (45)$$

The above equation means that the log-aesthetic space curve is given by a straight line in the $s - \sigma - \phi$ space where the three axes are the arc length s, σ and ϕ to connect two given points (s_1, σ_1, ϕ_1) and (s_2, σ_2, ϕ_2) as shown in Fig.3. In this case the following objective functional J_{LASC} is minimized.

$$\begin{aligned} J_{LASC} &= \int_{s_1}^{s_2} \sqrt{1 + \sigma_s^2 + \phi_s^2} ds \\ &= \int_{s_1}^{s_2} \sqrt{1 + \alpha^2 \rho^{2\alpha-2} \rho_s^2 + \beta^2 \mu^{2\beta-2} \mu_s^2} ds \quad (46) \end{aligned}$$

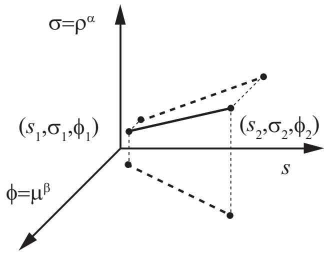

Figure 3: A straight line segment in the $s-\sigma-\phi$ space connecting two points (s_1, σ_1, ϕ_1) and (s_2, σ_2, ϕ_2).

The variational problem with n dependent variables is to minimize or maximize the following objective function

$$J_m = \int_{x_1}^{x_2} f(y_1, y_{1x}, y_2, y_{2x}, \cdots, y_n, y_{nx}, x) dx \quad (47)$$

where y_i $(i = 1, 2, \cdots, n)$ are functions of x and y_{ix} is a derivative of y_i with respect to x. y_i's are unknown. The Euler equations for the above functional are

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_{ix}} = 0 \quad (i = 1, 2, \cdots, n) \quad (48)$$

For Eq.(46), the Euler equations are given by Eq.(31) and the following equation

$$(\beta - 1)\mu_s^2 + \mu\mu_{ss} = 0 \quad (49)$$

5. CONCLUSIONS

We have reformulated the log-aesthetic curve by use of variational principle in order to analyze its properties. We have also derived a non-linear ordinary differential equation based on the variational formulation and justified the correctness of the variational formulation by solving it. Furthermore we have reformulated the log-aesthetic space curve and derived its non-linear ordinary differential equations in a similar manner.

As future work, we will define the log-aesthetic surface based on variational principle based on this research.

6. ACKNOWLEDGMENTS

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