Journal of Computational Design and Engineering, 2020, 7(0), 1–10

doi: 10.1093/jcde/qwaa014
Journal homepage: www.jcde.org

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RESEARCH ARTICLE

τ-curve: introduction of cusps to aesthetic curves☆

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☆Selected paper from the Asian Conference on Design and Digital Engineering, 1–3 November 2018, Okinawa, Japan.
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Abstract

Yan, Schiller, Wilensky, Carr, and Schaefer pointed out that one of the demerits of clothoid interpolation is a jumping behavior during the deformation of the curve. This phenomenon occurs because the clothoid curve cannot have a cusp, where the curve is kinked or the direction of the curve is abruptly changed. We discuss how to introduce cusps for the log-aesthetic curve including the clothoid curve and propose to use for the representation of a curve the direction angle instead of curvature and define a new curve named τ-curve, which is defined by the direction angle of the curve.

Keywords: log-aesthetic curve; σ-curve; τ-curve

1. Introduction

Yan, Schiller, Wilensky, Carr, and Schaefer (2017) proposed as κ-curve quadratic Bézier curve segments that pass through given points and that have minimum and maximum curvatures. Although the clothoid curve has been used in highway design and it is well studied how to connect them and circular arcs with G^2 continuity (Meek & Walton, 1989) and to interpolate G^2 Hermite data (Meek & Walton, 1998), they pointed out that the disadvantageous property of the clothoid curve (Cornu spiral), one of the log-aesthetic curves (Miura & Gobithaasan, 2016), is as follows: As shown in Fig. 1, if one of the control points of a piecewise clothoid curve is continuously moved, the curve suddenly flips unexpectedly. This is because in Fig. 2 when the tangent vector at the start point $\boldsymbol{P}_0$ is rotated, there are basically two cases: the curve is going in the left-top direction and reaches the end point $\boldsymbol{P}_1$ and the curve is going in the right-down direction and reaches $\boldsymbol{P}_1$ and that behavior is jumpy. In Fig. 2, the blue lines indicate the tangent directions of the curves at $\boldsymbol{P}_0$.

The log-aesthetic curve is an extension of the clothoid curve and it inherits the property of the clothoid curve mentioned earlier. For details regarding how to input the log-aesthetic curve, please refer to Yoshida and Saito (2006).

If the tangent of the curve is continuously changed, it is inevitable to have this type of the phenomenon since the curvature will jump from positive to negative, or vice versa. Therefore, to avoid this phenomenon, the curve should have a cusp, where the curve is kinked and the tangent of the curve is abruptly changed. In the next section, we will discuss about the cusp more mathematically.

Note that "aesthetic curves" in the title of this paper do not have any universally accepted mathematical definitions yet, but we regard them as curves that are suitable to be used in aesthetic design of industrial products. One of the most important properties of the aesthetic curves can be said to be the monotonicity of their curvature. Spiral curves have monotone curvature and have been an important tool in geometric modeling and computer-aided design. Bartoň and Elber (2011) introduced spiral fat arcs for bounding planar free-form curves more tightly than fat arcs. They take advantage of the curvature monotonicity of the spirals and their bounding region shows a cubic approximation order to a given spiral curve.

Received: 27 February 2019; **Revised:** 15 April 2019; **Accepted:** 9 November 2019

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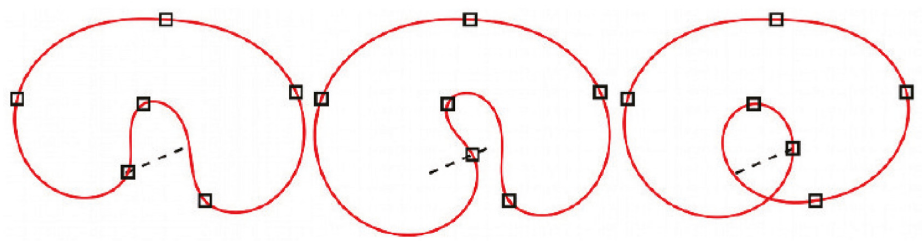

Figure 1: Flip of a piecewise clothoid curve (Yan et al., 2017).

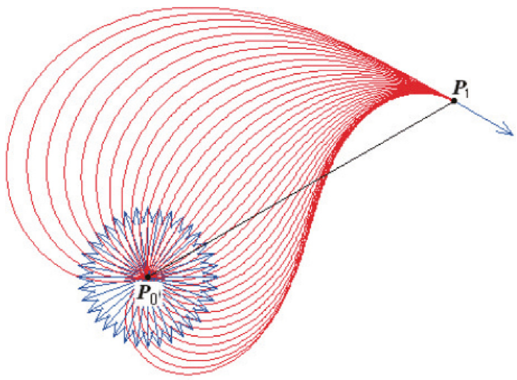

Figure 2: Shape changes according to the change of the tangent vector at one of the end points.

The curves with monotone curvature include clothoids (Meek & Walton, 1989), typical curves (Mineur, Lichah, Castelain, & Giaume, 1998), class A Bèzier curves (Farin, 2006), and the log-aesthetic curves, and they are frequently used for aesthetic design.

2. Cusp of the Plane Curve

The standard textbook of differential geometry on curves and surfaces (do Carmo, 1976) deals with a regular curve $\mathbf{C}(t)$, for which $d\mathbf{C}(t)/dt \neq 0$ for all t in its domain and no cusp is discussed in it. Textbooks on computer-aided geometric design (Cohen, Riesenfeld, & Elber, 2001; Farin, 2002; Hoscheck & Lasser, 1989) do not discuss about the cusp very well. The textbook written by Su and Liu (1989) analyzes cusps as well as inflection points of parametric cubic curves. A more general treatment on the cusp is presented by Porteous (2001).

In this section, based on Porteous (2001) we discuss cusps of a plane curve. A nonregular point is a point where the first derivative of the curve $d\mathbf{C}(t)/dt = 0$. Such a point is commonly called a cusp of the curve. A smooth, or infinitely differentiable, curve is said to have an ordinary, or 3/2, cusp at t if $d\mathbf{C}(t)/dt = 0$, but $d^2\mathbf{C}(t)/dt^2 \neq 0$ with $d^3\mathbf{C}(t)/dt^3$ linearly independent of $d^2\mathbf{C}(t)/dt^2$. Similarly, it is said to have an ordinary kink, or 4/3, cusp at t if $d\mathbf{C}(t)/dt = d^2\mathbf{C}(t)/dt^2 = 0$, but $d^3\mathbf{C}(t)/dt^3 \neq 0$ with $d^4\mathbf{C}(t)/dt^4$ linearly independent of $d^3\mathbf{C}(t)/dt^3$. More generally, $\mathbf{C}(t)$ is said to have an $(n+1)/n$ cusp at t if $d^i\mathbf{C}(t)/dt^i = 0$ for $1 \leq i < n$ but $d^n\mathbf{C}(t)/dt^n \neq 0$, with $d^{n+1}\mathbf{C}(t)/dt^{n+1}$ linearly independent of $d^n\mathbf{C}(t)/dt^n$. As we will explain in the next section, a cubic Bézier curve can have an ordinary cusp.

3. Cusps of Quadratic and Cubic Bézier Curves

3.1. Quadratic Bézier curve

Based on the above discussion, we analyze the properties of the cusp of a quadratic Bézier curve. At first, we locate its control points as shown in Fig. 3. Notice that the curve is symmetrical along the vertical line through the point at $t = 1/2$. Even if the distance between the first and second control points and that between the second and third control points are different, the similar situation can occur when the angle made by the control polyline at the second control point becomes 0. Suppose that the coordinates of the control points are $(-a, 0)$, $(0, 1)$, and $(a, 0)$, respectively. Gradually decreasing a, the curve becomes degenerated at $t = 1/2$. The curve $\mathbf{C}(t)$ degenerated at $t = 1/2$ is given by

$$\mathbf{C}(t) = (0, 2t(1-t)). \tag{1}$$

Their derivatives of degree from 1 to 3 are given by

$$\begin{aligned} \frac{d\mathbf{C}(t)}{dt} &= (0, 2(1-2t)), \\ \frac{d^2\mathbf{C}(t)}{dt^2} &= (0, -4), \\ \frac{d^3\mathbf{C}(t)}{dt^3} &= (0, 0). \end{aligned} \tag{2}$$

Hence,

$$\begin{aligned} \frac{d\mathbf{C}(1/2)}{dt} &= (0, 0), \\ \frac{d^2\mathbf{C}(1/2)}{dt^2} &= (0, -4), \\ \frac{d^3\mathbf{C}(1/2)}{dt^3} &= (0, 0). \end{aligned} \tag{3}$$

Since $d^3\mathbf{C}(t)/dt^3$ is linearly dependent on $d^2\mathbf{C}(t)/dt^2$, the curve does not have an ordinary cusp, but a commonly called cusp at $t = 1/2$.

The signed curvature of a plane curve $\mathbf{C}(t)$ is given by the following expression (do Carmo, 1976):

$$\kappa(t) = \frac{\mathbf{C}'(t) \times \mathbf{C}''(t)}{||\mathbf{C}'(t)||^3}, \tag{4}$$

where $\mathbf{C}'(t)$ and $\mathbf{C}''(t)$ represent the first and second derivatives with respect to t, respectively. $\boldsymbol{a} \times \boldsymbol{b}$ denotes the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $||\boldsymbol{v}||$ is the norm of vector $\boldsymbol{v}$. Applying these expressions to the quadratic Bézier curve defined earlier, we obtain

$$\kappa(t) = -\frac{a}{(a^2 + (1-2t)^2)^{3/2}}. \tag{5}$$

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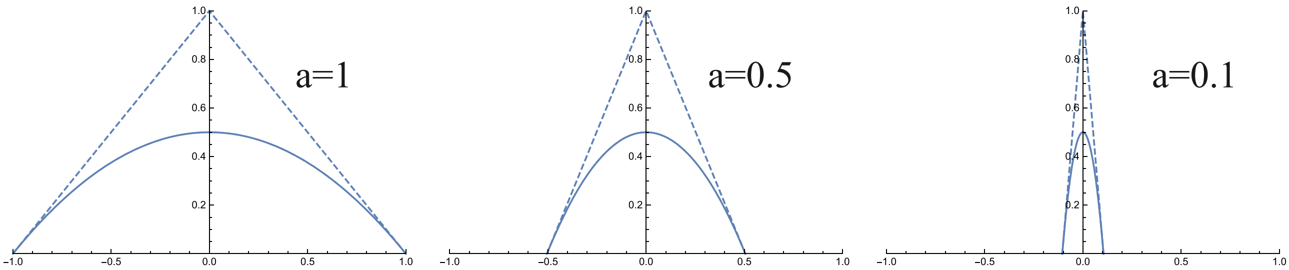

Figure 3: A quadratic Bézier curve.

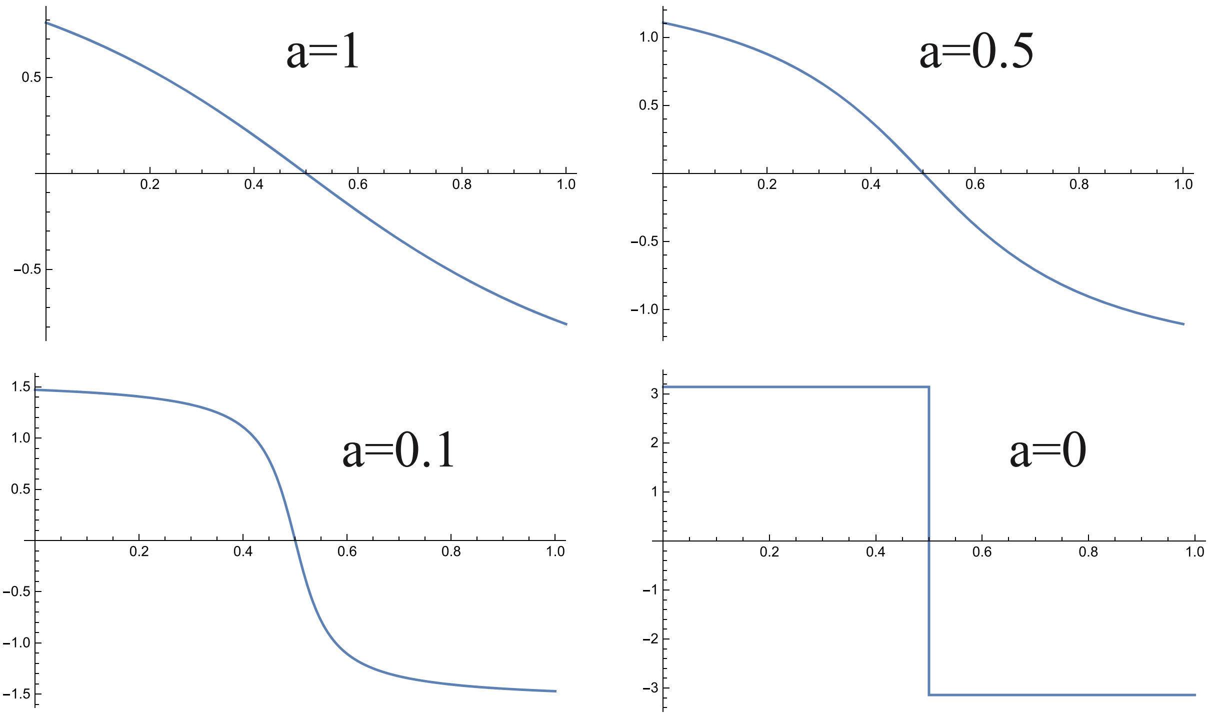

Figure 4: The direction angle of the quadratic Bézier curve.

Therefore, if $a \neq 0$, then $\kappa(t)$ is finite. When $t = 1/2$, then equation (5) becomes

$$\kappa\left(\frac{1}{2}\right) = -\frac{1}{a^2}. \tag{6}$$

When a approaches 0, $\kappa(1/2)$ become infinite (negatively infinite). Note that at first if we assume $a = 0$, then $\kappa(t) = 0$ except for $t = 1/2$. When t approaches 1/2, $\kappa(t) = 0/0$. We usually say that $\kappa(t)$ is not defined in this situation. Since we have two parameters a and t, the value of $\kappa(t)$ depends on how they approach (0, 1/2), but geometrically a cusp is generated at $t = 1/2$ as the analysis of the derivatives shows.

The direction angle $\theta(t)$ of the curve is given by

$$\theta(t) = \arctan\left(\frac{\Delta y}{\Delta x}\right) = \arctan\left(\frac{dy/dt}{dx/dt}\right) = \arctan\left(\frac{1-2t}{a}\right). \tag{7}$$

Figure 4 shows the graph of the direction angle of the quadratic Bézier curve. When $a \to 0$, the direction angle approaches a stair function, and for $0 \leq t < 1/2$, $\theta(t) = \pi/2$. When $t = 1/2$, $\theta(1/2) = \infty$, and for $1/2 \leq t \leq 1$, $\theta(t) = -\pi/2$. By using the direction angle, we can deal with a cusp naturally and no special treatment is necessary.

3.2. Cubic Bézier curve

In this section, we will analyze the properties of the cusp of a cubic Bézier curve. We arrange the positions of the control points as in Fig. 5. Note that the curve is symmetric along the vertical line through the point at $t = 1/2$. The coordinates of the control points are $(-1, 0)$, $(-a, 1)$, $(a, 1)$, and $(1, 0)$, respectively. Similarly to the quadratic Bézier curve case, we gradually decrease the value of a, make the curve degenerated, and cross the control points.

The curve $\mathbf{C}(t)$ degenerated at $t = 1/2$ is given by

$$\mathbf{C}(t) = ((2t-1)^3, 3(1-t)t). \tag{8}$$

Their derivatives of degree from 1 to 3 are given by

$$\begin{aligned}\frac{d\mathbf{C}(t)}{dt} &= (6(1-2t)^2, 3(1-2t)),\\ \frac{d^2\mathbf{C}(t)}{dt^2} &= (6(2t-1), -6),\\ \frac{d^3\mathbf{C}(t)}{dt^3} &= (12, 0).\end{aligned} \tag{9}$$

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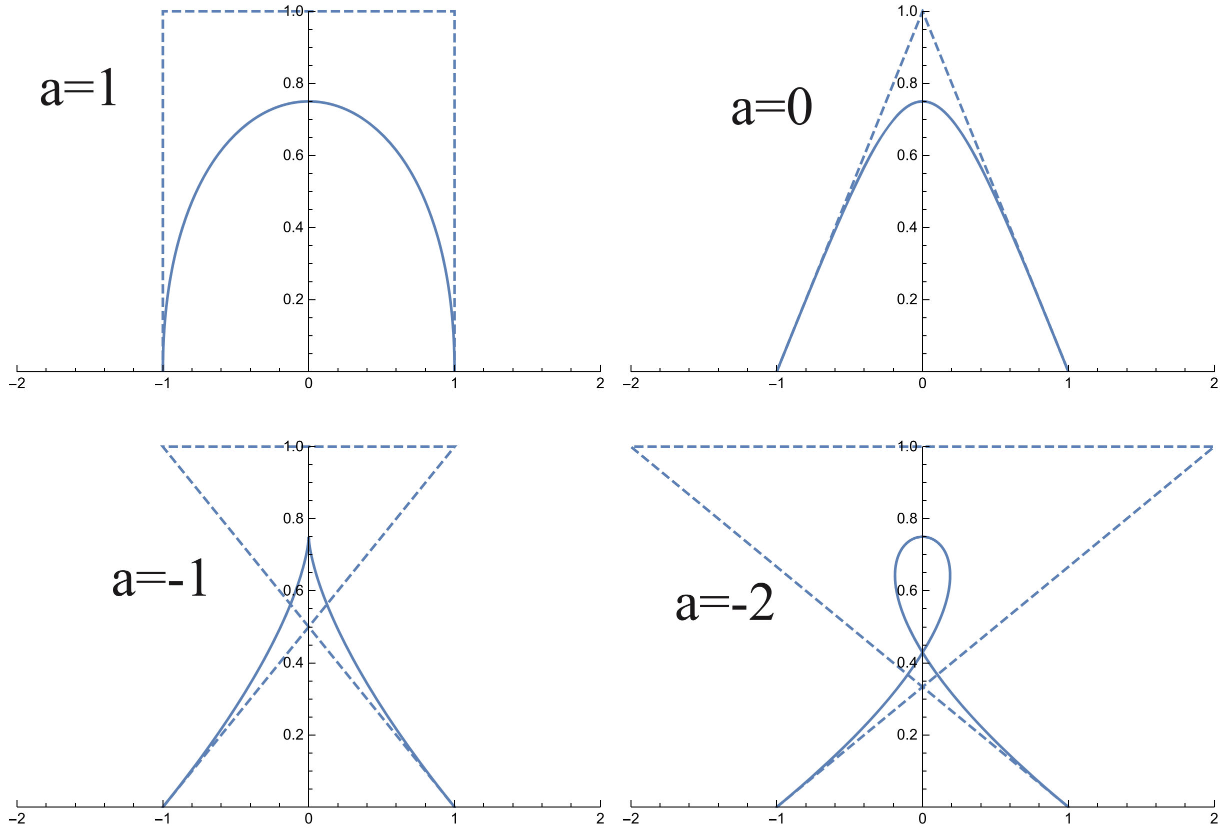

Figure 5: A cubic Bézier curve.

Hence,

$$\frac{d\mathbf{C}(1/2)}{dt} = (0, 0),$$
$$\frac{d^2\mathbf{C}(1/2)}{dt^2} = (0, -6),$$
$$\frac{d^3\mathbf{C}(1/2)}{dt^3} = (12, 0). \tag{10}$$

Since $d^4\mathbf{C}(t)/dt^4$ is linearly independent of $d^3\mathbf{C}(t)/dt^3$, the curve has an ordinary, 3/2, cusp at $t = 1/2$.

By using equation (4), the curvature of this cubic Bézier curve is given by

$$\kappa(t) = -\frac{4\left(3at^2-3at+a-t^2+t\right)}{3\left(a^2\left(6t^2-6t+1\right)^2-2a\left(12t^4-24t^3+20t^2-8t+1\right)+4t^4-8t^3+12t^2-8t+2\right)^{3/2}}. \tag{11}$$

Hence,

$$\kappa\left(\frac{1}{2}\right) = -\frac{8}{3\sqrt{(a+1)^2(a+1)}}. \tag{12}$$

From the above equation, when $a = -1$, the curvature becomes infinite (its sign depends on a's approaching direction). This corresponds to the case where $a = -1$ in Fig. 5, and when $t = 1/2$, a cusp is formed. Note that at first if we assume that $a = -1$, then

$$\kappa(t) = \frac{4}{3(1-2t)(5-16t+16t^2)^{3/2}}. \tag{13}$$

Thus, when t approaches 1/2, $\kappa(t)$ becomes really infinite. Geometrically, a cusp is generated as in the quadratic Bézier curve case.

The direction angle $\theta(t)$ of the curve is given by

$$\theta(t) = \arctan\left(\frac{2t-1}{2(3a-1)t^2-2(3a-1)t+a-1}\right). \tag{14}$$

Figure 6 shows the direction angle of the cubic Bézier curve. When $a = -1$, at $t = 1/2$ the direction angle reverses and its graph becomes similar to a sawtooth wave. When $a = -2$, the direction angle seems to jump, but this is because of its range and the direction angle itself changes smoothly. Even in this cubic case, we can manage to treat a cusp point naturally and no special process is required. Therefore, in order to introduce points with cusps for a curve, it is recommend to use the direction angle θ instead of the arc length s.

4. σ-curve

As a new type of the curve that keeps the merit (controllability of curvature) of the log-aesthetic curve and possesses another merit of symmetry, the σ-curve has been proposed (Miura et al., 2018). By using the Cesàro equation, a σ-curve is defined by

$$\mathrm{sgn}(\rho)|\rho|^{\alpha} = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \tag{15}$$

where sgn(·) is the sign function. More explicitly, if the right-hand side of the above equation is equal to 0 or positive, then $\rho \geq 0$ and

$$\rho^{\alpha} = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \tag{16}$$

else $\rho < 0$ and

$$(-\rho)^{\alpha} = -(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0). \tag{17}$$

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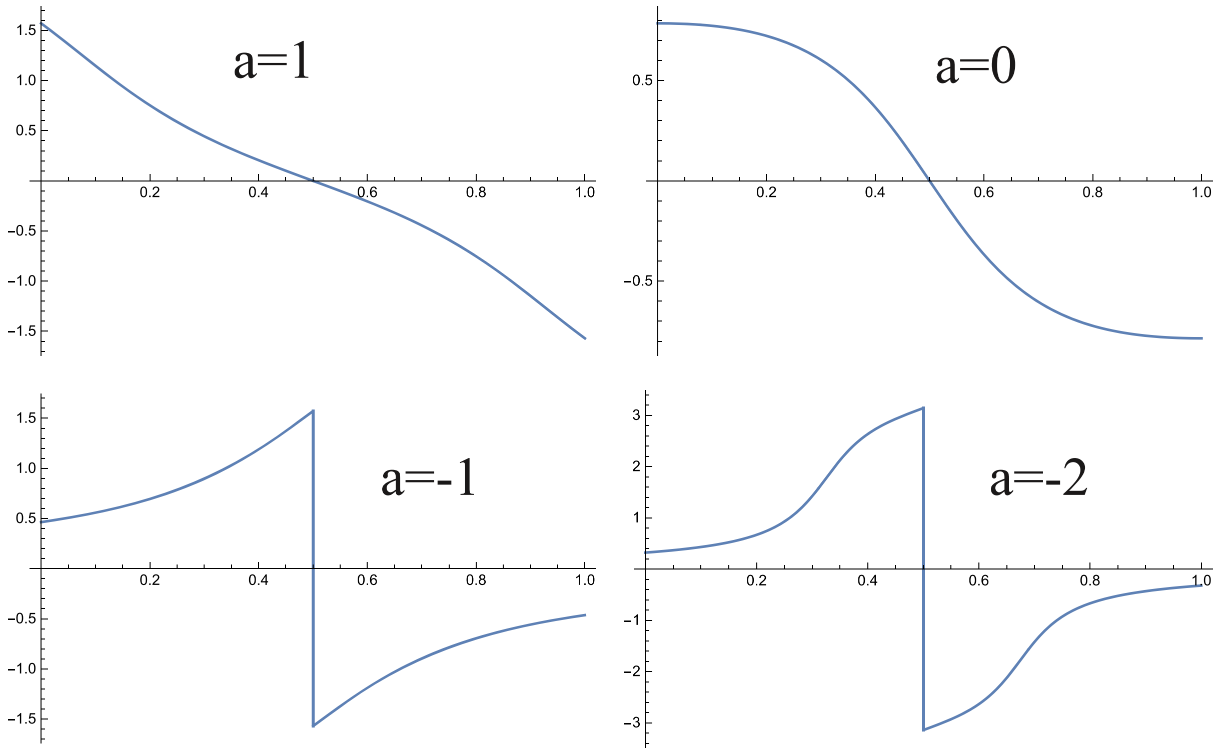

Figure 6: The angle of the cubic Bézier curve.

In these equations, $\sigma(s) = \rho(s)^\alpha$ is given by a polynomial function of arc length s. n is called the degree of the σ-curve. Note that in case of $n = 1$, it becomes a log-aesthetic curve (Miura & Gobithaasan, 2016). Suppose $\rho \geq 0$, then its curvature $\kappa(s)$ is given by

$$\kappa(s) = (a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)^{-1/\alpha}. \tag{18}$$

From the above equation and $d\theta(s)/ds = \kappa(s)$, when $n = 1$ and suppose the integral constant is equal to 0, direction angle $\theta_1(s)$ is given by the following expression and it can be integrated analytically:

$$\theta_1(s) = \frac{\alpha(a_1 s + a_0)^{(\alpha-1)/\alpha}}{(\alpha-1)a_1}. \tag{19}$$

5. τ-curve

According to the discussion so far, it is desirable to use direction angle $\theta(s)$ to introduce cusp points to a curve. Similarly to the σ-curve definition, the power β of $\theta(s)$ is given by a polynomial function of arc length s, i.e.

$$\mathrm{sgn}(\theta)|\theta|^\beta = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0. \tag{20}$$

Supposing $\theta \geq 0$,

$$\theta = \left\{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0\right\}^{1/\beta}. \tag{21}$$

In this research, the curve whose direction angle $\theta(s)$ is given in this way is called a τ-curve. Similarly to the σ-curve, n is called the degree of the τ-curve.

When $n = 1$,

$$\begin{aligned}\{\theta_1(s)\}^\beta &= b_1 s + b_0, \\ \theta_1(s) &= (b_1 s + b_0)^{1/\beta}.\end{aligned} \tag{22}$$

Suppose $\beta \neq 0, 1$, then the curvature $\kappa_1(s)$ of the curve is given by

$$\kappa_1(s) = \frac{d\theta(s)}{ds} = \frac{b_1}{\beta}(b_1 s + b_0)^{(1-\beta)/\beta} = \left(\frac{b_1^{1/(1-\beta)}}{\beta^{\beta/(1-\beta)}} s + \frac{b_1^{\beta/(1-\beta)} b_0}{\beta^{\beta/(1-\beta)}}\right)^{(1-\beta)/\beta}, \tag{23}$$

where $a_1 = b_1/\beta^{\beta/(1-\beta)}$ and $a_0 = b_1^{\beta/(1-\beta)} b_0/\beta^{\beta/(1-\beta)}$. Furthermore, suppose $\alpha = \beta/(\beta - 1)$, then we obtain

$$\kappa_1(s)^{-\alpha} = a_1 s + a_0. \tag{24}$$

When $\beta = 0, 1$, we can derive similar formulas. Therefore, a linear τ-curve is a log-aesthetic curve, as shown in Fig. 7.

Figure 8 shows several examples of quadratic τ-curves. All curves in the figure have the same total length and their direction angle is given by

$$\theta^\beta = \frac{1}{3}(s^2 - 4s + 3) = \frac{1}{3}(s-1)(s-3) \tag{25}$$

to clarify the effects of different β values on their shape. We changed β from 0.25 to 1.5 in increments of 0.25.

5.1. Curvature of τ-curve of degree n

From equation (21), if we assume that $\theta \geq 0$ and $\beta \neq 0, 1$, then the curvature $\kappa(s)$ of the curve is

$$\begin{aligned}\kappa(s) = \frac{1}{\beta}&\left\{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0\right\}^{(1-\beta)/\beta} \\ &\times (nb_n s^{n-1} + (n-1)b_{n-1}s^{n-2} + \cdots + b_1).\end{aligned} \tag{26}$$

In case of $n = 2$,

$$\kappa_2(s) = \frac{1}{\beta}\left\{b_2 s^2 + b_1 s + b_0\right\}^{(1-\beta)/\beta}(2b_2 s + b_1). \tag{27}$$

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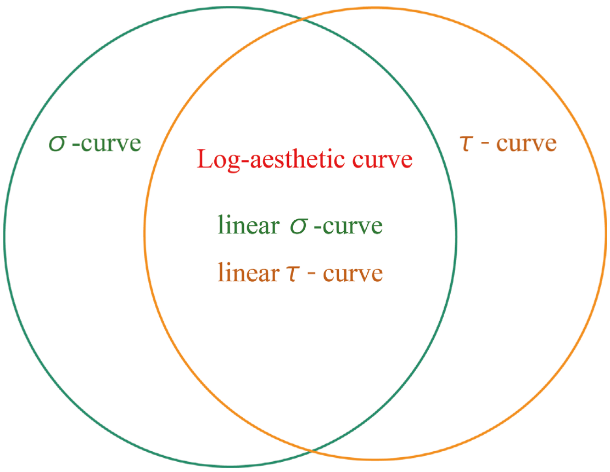

Figure 7: Relationship between σ-curve and τ-curve.

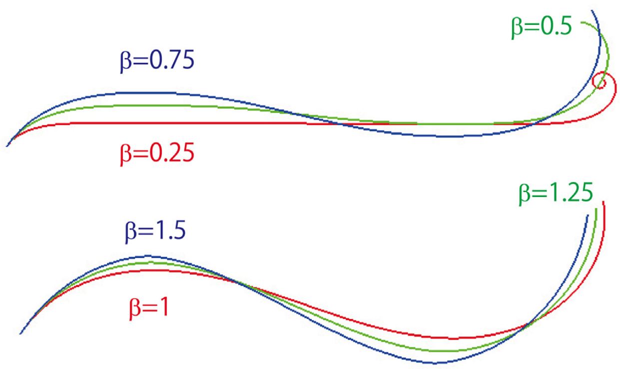

Figure 8: Quadratic τ curve: length = 4.5, $\theta^\beta = (s^2 - 4s + 3)/3 = (s - 1)(s - 3)/3$.

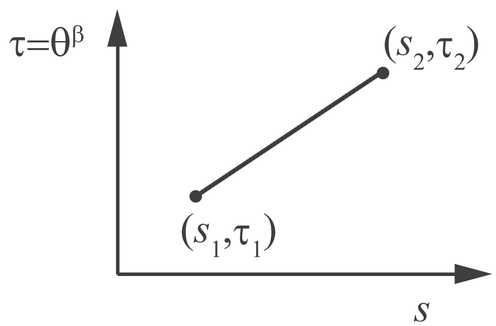

Figure 9: s–τ space.

In case of $\beta = 1/2$,

$$\kappa_2(s) = 2(b_2 s^2 + b_1 s + b_0)(2b_2 s + b_1) \tag{28}$$

and curvature $\kappa_2(s)$ is a cubic function of s.

5.2. The formulation of τ-curve based on variational principle

5.2.1. *Linear curve*

Like a linear σ-curve is given by a straight line segment in aesthetic space (s–σ space), a linear τ-curve is given by a straight line segment in s–τ space (please refer to Fig. 9).

Suzuki, Gibithaasan, Salvi, Usuki, and Miura (2018) defined a functional that is minimized by a log-aesthetic curve with total length l as follows:

$$K_{LAC} = \int_0^l (\sigma_s)^2\, ds. \tag{29}$$

By corresponding σ with τ, we obtain the following equation:

$$\begin{aligned} K_\tau &= \int_0^l (\tau_s)^2\, ds = \int_0^l \left(\beta\theta^{\beta-1}\frac{d\theta}{ds}\right)^2 ds \\ &= \int_0^l (\beta\theta^{\beta-1}\kappa)^2\, ds = \beta^2 \int_0^l (\theta^{\beta-1}\kappa)^2\, ds, \end{aligned} \tag{30}$$

where from $\alpha = \beta/(\beta - 1)$, $\beta = \alpha/(\alpha - 1)$ and the above equation is rewritten by

$$K_\tau = \frac{\alpha^2}{(\alpha-1)^2} \int_0^l (\theta^{1/(\alpha-1)}\kappa)^2\, ds. \tag{31}$$

Coefficient $\alpha^2/(\alpha - 1)^2$ is not necessary because if we fix α, it does not affect the extremum function, so we get

$$K_\tau = \int_0^l (\theta^{1/(\alpha-1)}\kappa)^2\, ds. \tag{32}$$

5.2.2. *Quadratic curve*

For a quadratic τ-curve,

$$\tau_{sss} = 0. \tag{33}$$

Hence, it minimizes

$$\begin{aligned} K_{q\tau} &= \int_0^l (\tau_{ss})^2\, ds = \int_0^l (\beta(\beta-1)\theta^{\beta-2}\kappa^2 + \beta\theta^{\beta-1}\kappa_s)^2\, ds \\ &= \beta^2 \int_0^l \theta^{2\beta-4}((\beta-1)\kappa^2 + \theta\,\kappa_s)^2\, ds. \end{aligned} \tag{34}$$

We can omit coefficient β^2 and

$$K_{q\tau} = \int_0^l \theta^{2\beta-4}((\beta-1)\kappa^2 + \theta\,\kappa_s)^2\, ds. \tag{35}$$

5.3. τ-curve with cusps

In order to solve the original problem of adding cusp points to τ-curves, we extend equation (20) as follows:

$$\mathrm{sgn}(\theta)|\theta|^\beta = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 + \pi \sum_{i=1}^{n} U(s - s_i), \tag{36}$$

where $U(x)$ is a unit step function and is defined by

$$U(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{37}$$

By adding the terms of the step functions, the direction angle jumps by π at s_i, inducing a cusp. Figure 10 shows a simple example of a curve with a cusp. Since $s_1 = 1$, the curve has a cusp where $s = 1$. Note that the change of the direction angle at a cusp is equal to $\pm\pi$ and we can use $\pm U(s - s_i)$ instead of $U(s - s_i)$ in equation (36). Furthermore, although the change of the direction angle at the cusp is $\pm\pi$, we can specify an arbitrary angle instead of π in equation (36).

5.4. Curve generation by minimizing functionals

We intend to generate a curve approximating a log-aesthetic curve and we minimize equation (32) to generate a curve. By specifying the start and end points and the tangent vectors

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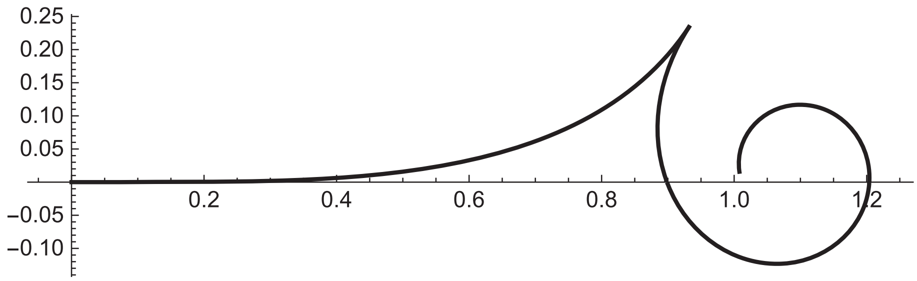

Figure 10: A τ-curve example with a cusp point: $\theta^{1/3} = s$, $n = 1$, $s_1 = 1$, $0 \le s \le 2$.

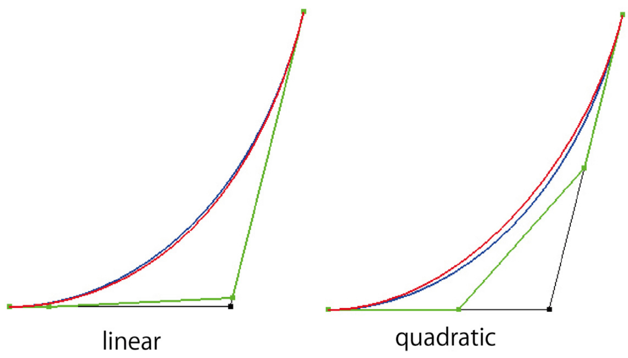

Figure 11: A log-aesthetic curve in blue and generated τ-curves by optimization in red.

there, we generate a cubic Bézier curve. Hence, by changing the lengths between the first and second points and the third and fourth points and minimizing the functional, we determine the locations of the second and third control points.

We expect the following issues for this optimization:

(i) The same shaped curve should be generated even if the boundary condition is rotated about an arbitrary point and the direction angles of the tangent vectors have a degree of freedom of rotation. Hence, by changing direction angle from θ to $\theta + \theta_d$, where θ_d is an arbitrary value, the same shaped curve should be generated by minimizing equation (32).

(ii) When $\beta - 1 < 0$, i.e. $\beta < 1$, so $1/(\alpha - 1) < 0$, i.e. $\alpha < 1$, for direction angle $\theta \le 0$, $\theta^{\beta-1}$ becomes infinite or undetermined (a complex number). We have to avoid this situation.

At first, we attack the first issue. We change θ by rotation angle θ_d. From equation (21),

$$\theta + \theta_d = (b_1 s + b_0)^{1/\beta}. \tag{38}$$

Hence,

$$\theta + \theta_d = (b_1 s + b_0)^{1/\beta}. \tag{39}$$

Therefore,

$$(\theta + \theta_d)^\beta = b_1 s + b_0. \tag{40}$$

The above equation means that by minimizing functional (32) of $\tau' = (\theta + \theta_d)^\beta$, we obtain an approximation curve of a linear τ-curve, i.e. a log-aesthetic curve. Equation (32) is defined by

$$K_\tau = \int_0^l \{(\theta + \theta_d)^{1/(\alpha-1)}\kappa\}^2 \, ds. \tag{41}$$

Note that θ_d is an arbitrary angle.

As a matter of fact, that the direction angle changes by θ_d solves the second issue. We add a large angle θ_d to avoid the situation where $\theta < 0$. For example, by adding 10 rad, we can avoid a negative θ. The important point for implementation is when θ is relatively small, in case where $\beta - 1$ is negative and its absolute value is large, $\theta^{\beta-1}$ could be a very large and it causes large errors and we cannot generate a suitable curve.

Figure 11 shows examples of optimization. On left- and right-hand sides of the figure, the curve in blue is a log-aesthetic curve satisfying the given boundary condition whose $\alpha = -1/2$ and the curve in red is a cubic Bézier curve generated by subdividing the tangent vectors determined by the three points by 1000 and minimizing functional (35) for the linear curve on the left-hand side and functional (41) for the quadratic curve on the right-hand side. The polyline in green connects the control points of the cubic curve. By minimization, we obtain a good approximation curve with the log-aesthetic curve. For both the curves, $\beta = \alpha/(\alpha - 1) = 1/3$. We add $\theta_d = 4\pi$ rad to θ.

The linear curve is almost the same as the log-aesthetic curve and the quadratic curve has a smaller curvature than that of the log-aesthetic curve. We will investigate the effect of θ_d for given boundary conditions.

5.5. G^1 Hermite interpolation by τ-curve with cusps

In the following two sections, we will show two applications of the τ-curve to validate its usage for practical purposes. The objective of this paper is basically to produce the τ-curve and we do not intend to extract full potential of the curve formulation.

As one of the application examples of τ-curve, we implemented Matlab® codes to extract outlines of a Chinese character. The original Chinese characters stand for the name of the new era, "reiwa" in Japan, and we selected two strokes of the first Chinese character. It is very common for multiple strokes to overlap each other. We used fonts, but the codes are programmed to be able to process manually written characters.

At first, the outline was extracted as a sequence of points and they were smoothed by the moving average filter. Then, the discrete curvatures of the outline were calculated and the points where their curvatures are larger than a given threshold were detected as cusps. The number of detected cusps is 10, as shown in the top-right part in Fig. 12. The middle-right part in Fig. 12 shows the line segments connecting the cusps and the points with curvatures higher than the thresholds. Each segment separated by the cusps and the high-curvature points were subdivided to make sure for each subdivided segment to have monotonic curvature. Finally, by using the positions of the start and end points and the tangents there, a discrete linear τ-curve was generated to satisfy G^1 continuity except for the cusps and the high-curvature points, where only G^0 is guaranteed. Note that

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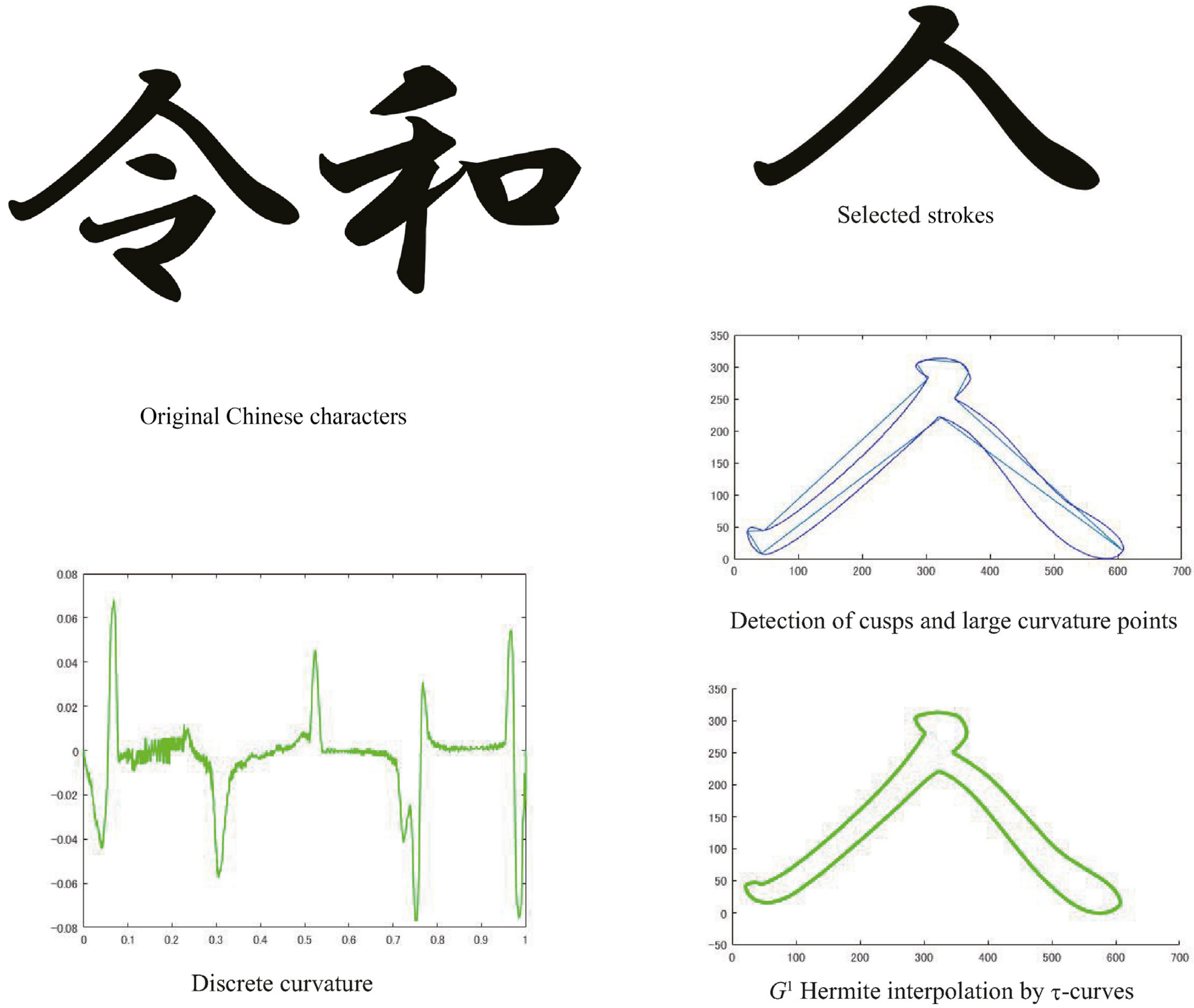

Figure 12: G^1 Hermite interpolation by τ-curve with cusps.

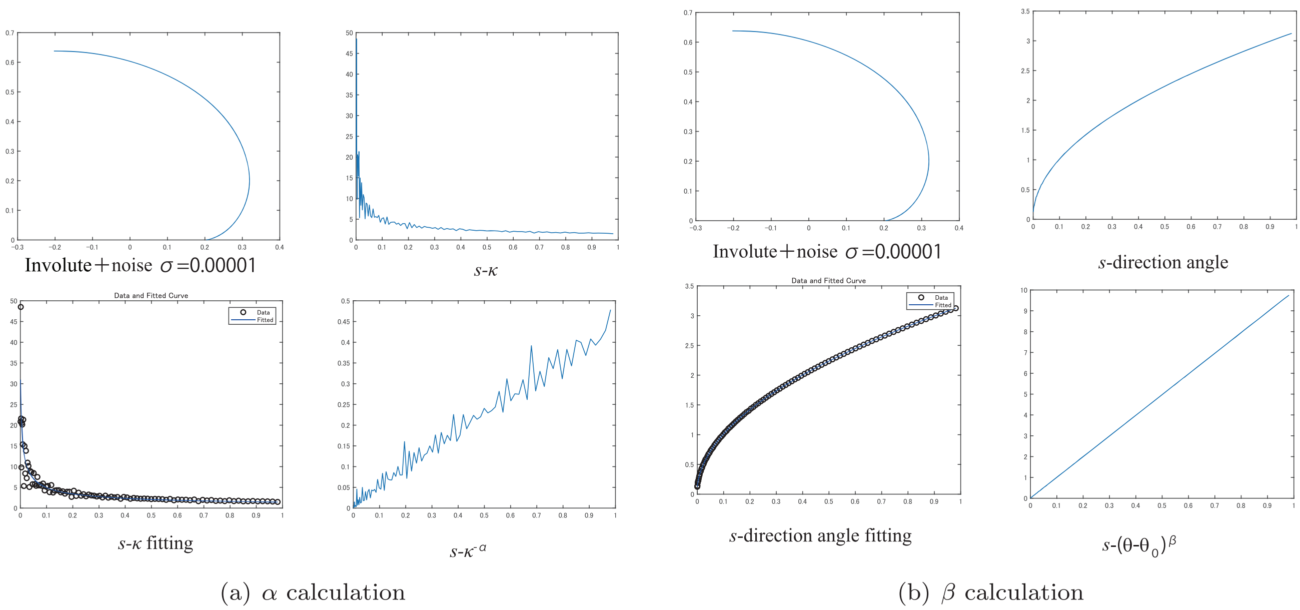

Figure 13: (a) α calculation and (b) β calculation for a circle involute with weaker noise: $\sigma = 0.00001$.

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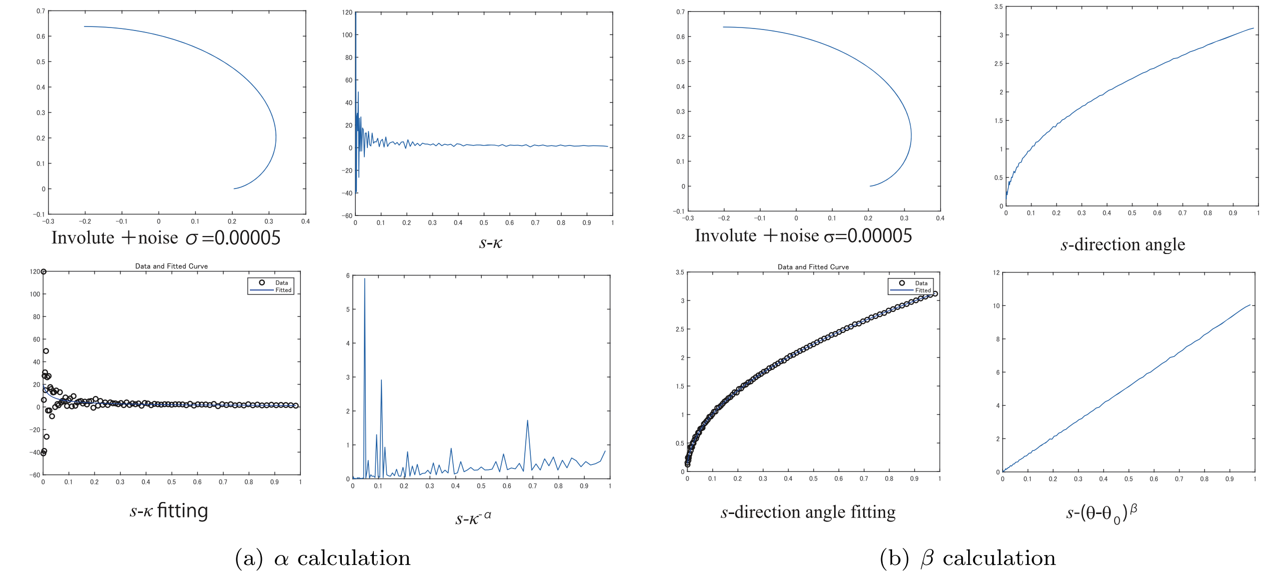

Figure 14: (a) α calculation and (b) β calculation for a circle involute with stronger noise: $\sigma = 0.00005$.

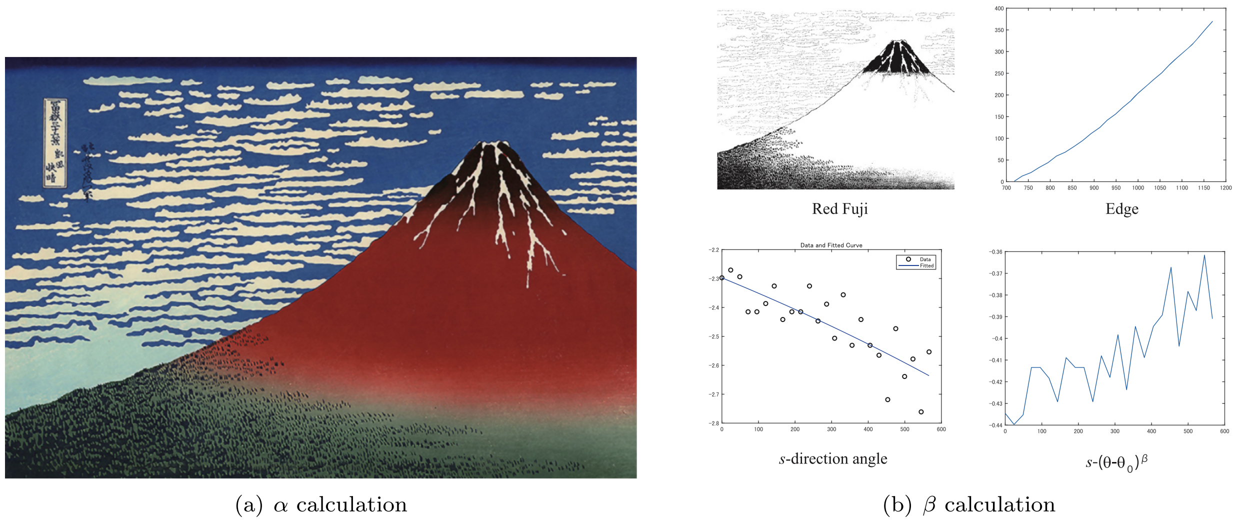

Figure 15: (a) α calculation and (b) β calculation for Aka-Fuji (Red Fuji).

the linear τ-curve is equivalent to the log-aesthetic curve and their discrete version is generated by the method described in Inoguchi, Kajiwara, Miura, Park, and Schief (2018). We used parameter $\beta = 1/3$ ($\alpha = -1/2$) for every τ-curve. The generated τ-curves approximate the outline very well, as shown in the bottom-right figure.

5.6. Estimation of shape parameters α and β

The shape parameter of the log-aesthetic curve has a strong effect of the impression of the curves (Harada, 1997). In this section, we discuss about how to calculate of the shape parameter β of the linear τ-curve instead of α of the log-aesthetic curve.

We utilized a built-in function `lsqcurvefit()` in Matlab®, which performs least-square fitting, to estimate α and β. For α, we used the Cesáro equation of the log-aesthetic curve, i.e. a linear σ-curve, as follows:

$$\kappa = \begin{cases} (c\,s + d)^{-1/\alpha}, & c\,s + d \geq 0, \\ -(-c\,s - d)^{-1/\alpha}, & \text{otherwise.} \end{cases} \tag{42}$$

In the above equation, the variables to be determined are α, c, and d. We input arc length s and κ of a unit-length circle involute curve with noise of standard deviation $\sigma = 0.00001$ and calculated them as shown in Fig. 13a. The calculated values are $\alpha = 1.841$, $c = 0.5131$, $d = 0.001419$, and root mean square (RMS) = 10.4.

For β, we used the following equation of the linear τ-curve:

$$\theta = \begin{cases} (c\,s + d)^{1/\beta} + e, & c\,s + d \geq 0, \\ -(-c\,s - d)^{1/\alpha} + e, & \text{otherwise.} \end{cases} \tag{43}$$

In the above equation, the variables to be determined are β, c, d, and e. We input arc length s and direction angle θ instead of κ

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of the same curve used for α calculation, as shown in Fig. 13b. The calculated values are $\beta = 2.004$, $c = 9.927$, $d = 0.01232$, $e = 0.01016$, and RMS $= 0.003113$.

Since we used a circle involute, both α and β should be 2. Even with noise, β is very close to 2 because we use the first derivative related value of the curve, i.e. θ, instead of the second derivative related value of the curve, i.e. κ.

As shown in Fig. 14 for a stronger noise, i.e. $\sigma = 0.00005$, the variables of the log-aesthetic curve were $\alpha = 1.564$, $c = 0.6647$, $d = 0.009096$, and RMS $= 60.55$. Those of the linear τ-curve were $\beta = 2.022$, $c = 9.927$, $d = 0.01232$, $e = 0.01016$, and RMS $= 10.4$. We could obtain β much closer to 2.

In Fig. 15, we show the preliminary result of the calculation of β for the mountain left ridgeline of Aka-Fuji (Red Fuji), which is a famous ukiyoe. The variables of the linear τ-curve were $\beta = -1.001$, $c = 0.00009857$, $d = -0.4346$, $e = 0.001353$, and RMS $= 0.01553$. We will continue our research to calculate β for various industrial products and artistic works.

6. Conclusions and Future Work

The purpose of the paper is to propose a new type of free curves for computer-aided design and other computer-related communities, especially for aesthetic design of industrial parts. For practical design, Bézier, B-spline, and NURBS are dominant choices, and in spite of many researchers' efforts, people are using the traditional representations of free-form curves and surfaces. The curve we have proposed in this paper is a new type of aesthetic curve, which includes the log-aesthetic curve when the curve is of degree 1. The parameter of the τ-curve is direction angle instead of curvature to eliminate the demerit of clothoid interpolation, which is a jumping behavior during the deformation of the curve. We have discussed about its properties and formulation-based variational principle, and shown various curve examples. One of the theoretical advantages of the new curve is that we are able to find a good method to estimate shape parameter α of the log-aesthetic curve because the degree of the derivatives of the τ-curve for its definition is less than that of the log-aesthetic curve; i.e. the τ-curve is defined by the first derivative: θ instead of radius of curvature, which is related to the second derivative of the curve. We have also shown that the shape parameter β can be determined by the direction angles only and stably calculated.

For future work, we will investigate the effect of θ_d for given boundary conditions and what practical situations are suitable to use cusps of τ-curves and to use quadratic ones.

Acknowlegements

This work was supported in part by JSPS Grant-in-Aid for Scientific Research (B) grant number 19H02048, JSPS Grant-in-Aid for Challenging Exploratory Research grant number 26630038, JST RISTEX Service Science, Solutions and Foundation Integrated Research Program, ImPACT Program of the Council for Science, Technology and Innovation, and FRGS grant (59431) provided by Universiti Malaysia Terengganu and Ministry of Education, Malaysia.

Conflict of interest statement

Declarations of interest: none.

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